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Qualitative Calculi with Heterogeneous Universes

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Abstract

Qualitative representation and reasoning operate with non-numerical relations holding between objects of some universe. The general formalisms developed in this field are based on various kinds of algebras of relations, such as Tarskian relation algebras. All these formalisms, which are called qualitative calculi, share an implicit assumption that the universe is homogeneous, i.e., consists of objects of the same kind. However, objects of different kinds may also entertain relations. The state of the art of qualitative reasoning does not offer a general combination operation of qualitative calculi for different kinds of objects into a single calculus.

Many applications discriminate between different kinds of objects. For example, some spatial models discriminate between regions, lines and points, and different relations are used for each kind of objects. In ontology matching, qualitative calculi were shown useful for expressing alignments between only one kind of entities, such as concepts or individuals. However, relations between individuals and concepts, which impose additional constraints, are not exploited.

This dissertation introduces modularity in qualitative calculi and provides a methodology for modeling qualitative calculi with heterogeneous universes. Our central contribution is a framework based on a special class of partition schemes which we call modular. For a qualitative calculus generated by a modular partition scheme, we define a structure that associates each relation symbol with an abstract domain and codomain from a Boolean lattice of sorts. A module of such a qualitative calculus is a sub-calculus restricted to a given sort, which is obtained through an operation called relativization to a sort. Of a greater practical interest is the opposite operation, which allows for combining several qualitative calculi into a single calculus. We define an operation called combination modulo glue, which combines two or more qualitative calculi over different universes, provided some glue relations between these universes. The framework is general enough to support most known qualitative spatio-temporal calculi.

Résumé

Représentation et raisonnement qualitatifs fonctionnent avec des relations non-numériques entre les objets d'un univers. Les formalismes généraux développés dans ce domaine sont basés sur différents types d'algèbres de relations, comme les algèbres de Tarski. Tous ces formalismes, qui sont appelés des calculs qualitatifs, partagent l'hypothèse implicite que l'univers est homogène, c'est-à-dire qu'il se compose d'objets de même nature. Toutefois, les objets de différents types peuvent aussi entretenir des relations. L'état de l'art du raisonnement qualitatif ne permet pas de combiner les calculs qualitatifs pour les différents types d'objets en un seul calcul.

De nombreuses applications discriminent entre différents types d'objets. Par exemple, certains modèles spatiaux discriminent entre les régions, les lignes et les points, et différentes relations sont utilisées pour chaque type d'objets. Dans l'alignement d'ontologies, les calculs qualitatifs sont utiles pour exprimer des alignements entre un seul type d'entités, telles que des concepts ou des individus. Cependant, les relations entre les individus et les concepts, qui imposent des contraintes supplémentaires, ne sont pas exploitées.

Cette thèse introduit la modularité dans les calculs qualitatifs et fournit une méthodologie pour la modélisation de calculs qualitatifs des univers hétérogènes. Notre contribution principale est un cadre basé sur une classe spéciale de schémas de partition que nous appelons modulaires. Pour un calcul qualitatif engendré par un schéma de partition modulaire, nous définissons une structure qui associe chaque symbole de relation avec un domaine et codomain abstrait à partir d'un treillis booléen de sortes. Un module d'un tel calcul qualitatif est un sous-calcul limité à une sorte donnée, qui est obtenu par une opération appelée relativisation à une sorte. D'un intérêt pratique plus grand est l'opération inverse, qui permet de combiner plusieurs calculs qualitatifs en un seul calcul. Nous définissons une opération appelée combinaison modulo liaison, qui combine deux ou plusieurs calculs qualitatifs sur différents univers, en fonction de relations de liaison entre ces univers. Le cadre est suffisamment général pour soutenir la plupart des calculs spatio-temporels qualitatifs connus.

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Chapter 1

Introduction

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1.1 Qualitative calculi

Qualitative knowledge may be expressed in terms of non-numerical (qualitative) relations holding between some entities (objects). Qualitative relations are used in natural language when we speak about space, time and other commonsense concepts. *Qualitative reasoning* deals with qualitative knowledge expressed by a limited vocabulary of relations. Typically, such a family of relations is logically constrained in that certain combinations of relations are possible whilst others are impossible. These logical dependencies between relations can be captured algebraically. This is the essence of the algebraic approach to qualitative reasoning (hereafter *algebraic reasoning*). Algebraic reasoning treats relations as first-order citizens (primitives, *symbols*) and concentrates the useful information about relations within relational operations.

Qualitative reasoning is traditionally studied in the context of reasoning about time and space. The notion of a *qualitative calculus* is central in the inquiry of qualitative spatio-temporal reasoning. A qualitative calculus consists of two parts: an algebra of relations and an interpretation structure. The *algebra of relations* serves both for representation of relations and for reasoning about relational facts. The interpretation structure defines the semantics of relation symbols. A classical example of a qualitative calculus is Allen's interval calculus. A qualitative calculus can be characterized in terms of the complexity and algorithmic properties of the algebra of relations with respect to the semantics of relations.

The term "qualitative calculus" is used in two different senses: to refer to a particular algebra of relations equipped with an interpretation structure, or to refer to a class of algebras of relations with a certain class of interpretation structures. An example of the latter is the qualitative calculus of Ligozat and Renz (2004), defined as a *non-associative (relation) algebra* weakly represented over some universe. Here, “qualitative calculus” is a *framework*, within which the properties of qualitative calculi are studied in a systematic way. However, there is no common agreement on the definition of a qualitative calculus. Some different frameworks are proposed in Nebel and Scivos (2002), Dylla et al. (2013), Westphal et al. (2014).

All frameworks for qualitative calculus assume a concrete universe. Relation symbols are interpreted as binary relations over the universe. With such interpretations, reasoning tasks can be formulated as a binary *constraint-satisfaction problem* (binary CSP, or BCSP), with a *relational structure* as a parameter. Thus, a qualitative calculus can be seen as a framework for applying algebraic reasoning to an instance of BCSP.

1.2 Motivation

Since Allen’s pioneering work, many spatio-temporal calculi have been developed. Each calculus covers some particular aspect of its domain. For example, some important aspects of the spatial domain are topology, orientation and distance. The problem of combining qualitative calculi, i.e., using them in a complementary way, is of actual importance. Qualitative calculi can be combined either algorithmically or structurally. The structural approach implies an operation which combines the candidate calculi into a single calculus.

Structural combination of qualitative calculi involves two levels: the syntactic level and the semantic level. On the semantic level, one has to take into account two “dimensions” of calculi: its entity model and relational model. For example, in the Region Connection Calculus, the entities are modeled as regular closed subsets of some topological space. The entities of the Rectangle Calculus are rectangles of a Euclidean space. The combined entity model would be *heterogeneous*, which means that it would consist of entities of different kinds. The combined relational model, apart from the already present relations, could be augmented with relations holding between entities of different kinds. On the syntactic level, the challenge lies in the fact that algebras of relations do not provide means for representing *kinds* of entities.

One of the approaches of qualitative reasoning is to formulate dependencies between relations as an axiomatic theory in some logical language. For example, in Randell and Cohn (1989), Eschenbach and Kulik (1997) some spatial relations are captured using first-order logic. If the domain theory is conjoined with a set of relational facts, consequences of these facts can be determined using any proof procedure which is complete for that language. From a computational point of view, this approach is unsatisfactory for all but the simplest sets of relations. Reasoning with a sufficiently expressive general-purpose logic is in most cases undecidable and at best an NP-complete problem.

There are special cases when the algebraic approach is applicable for reasoning about relational facts. There exists a methodology for generating an algebra based on a family of jointly exhaustive and pairwise disjoint predicates. When theories defining two families of such predicates are combined, one would like to combine the corresponding algebraic calculi. In qualitative reasoning, there is no methodology for doing this.

The original motivation of this thesis was to establish a formal framework for applying algebraic reasonig in ontology alignments. An alignment is a set of correspondences between two ontologies. A correspondence is an assertion that a certain relation holds between two ontological entities. It is expressed using a limited vocabulary of relations, such as "is a", "subsumed by", "part of", "has active ingredient", "may treat", etc. Two main problems for applying algebraic reasoning in alignments are that the universe is heterogeneous and that the semantics of alignment relations is not concrete. Since these problems are relevant in a broader scope of qualitative reasoning, we formulate our research questions in a general manner:

1. How to combine qualitative calculi with relations defined over different universes, given some "glue" relations between these universes?

2. How to combine qualitative calculi with relations defined within an axiomatic theory?

1.3 Contributions

We define non-associative partition schemes and show that there is a one-to-one correspondence between semi-strong representations (Mossakowski et al., 2006) and non-associative partition schemes. Every finite constraint language can be embedded into a non-associative partition scheme, which is not true for partition schemes of Ligozat and Renz. We show that every finite constraint language has (at least) a semi-associative qualitative calculus.

Our central contribution is a framework which introduces modularity in qualitative calculi. The framework is based on a special class of partition schemes, which we call modular. For a qualitative calculus generated by a modular partition scheme, we define a structure that associates each relation symbol with an abstract domain and codomain from a Boolean lattice of sorts. A module of such qualitative calculi is a sub-calculus restricted to a given sort, which is obtained through an operation called relativization to a sort. Of a greater practical interest is the opposite operation, which allows for combining several qualitative calculi into a single calculus. We define an operation called combination modulo glue, which combines two or more qualitative calculi over different universes, provided some glue relations between these universes. The framework is general enough to support all known qualitative spatio-temporal calculi.

We apply the developed theory to ontology alignments and define a modular qualitative calculus $\mathbb{A}16$, which covers taxonomical relations between individuals and classes. It improves on the calculus $\mathbb{A}5$ considered in Euzenat (2008) in two ways. First, $\mathbb{A}16$ combines class-level and instance-level relations within a single calculus. Second, it allows for discriminating between unsatisfiability and incoherence of alignments.

Ontology alignments are often equipped with numerical attributes, which express the confidence of each correspondence. We define a relaxed semantics of confidence values for subsumption and equivalence relations. We introduce a quasi-qualitative calculus $\mathbb{A}_{\mathsf{INTREC}}$ with infinitely many numerically parametrized relations, which can be used for expressing and reasoning with weighted relations in compliance with their relaxed semantics.

1.4 Outline of the dissertation

The rest of the dissertation is structured as follows. In Chapter 2, we give a general account of algebraic theories used in the dissertation. Chapter 3 provides the state of the art in algebraic frameworks for qualitative reasoning. Chapter 4 covers the state of the art in algebraic reasoning applied to ontology alignments. In Chapter 5, we provide a methodology for generating a qualitative calculus for any finite constraint language. Chapter 6 introduces the class of modular qualitative calculi and studies their properties. Chapter 7 applies the developed theory to ontology alignments. We introduce a novel qualitative calculus $\mathbb{A}16$, which consists of ontology alignment relations between individuals and classes. In Chapter 8, we introduce an infinite quasi-qualitative calculus $\mathbb{A}_{\mathsf{INTREC}}$, which can express ontology alignment relations with relaxed semantics.

Part I

Preliminaries

Chapter 2

Algebraic background

In this chapter, we give the algebraic machinery that will be used throughout the dissertation.

2.1 Notation and basic definitions

A *binary relation* over a set U is a subset of the Cartesian product $U \times U$. The *converse* (also called *inverse*, or *strong converse*) of a binary relation R is a relation symmetric to R, defined as $R^{-1} = \{(x, y) : (y, x) \in R\}$. The relation $Id_U = \{(x, x) : x \in U\}$ is called the *identity relation* over U. *Composition* of binary relations R and S is defined as $R \circ S = \{(x, y) : \exists z\ ((x, z) \in R \wedge (z, y) \in S)\}$. The *domain* and *codomain* of a binary relation R are defined as $Dom(R) = \{x : \exists y\ ((x, y) \in R)\}$ and $Cod(R) = \{y : \exists x\ ((x, y) \in R)\}$. The *field* of R is the union if its domain and codomain: $Fd(R) = Dom(R) \cup Cod(R)$. The field of R is also the smallest among all sets U for which $R \subseteq U \times U$. Let $\mathcal{B} = (R_i)_{i \in I}$ be a set of binary relations. By $\mathcal{B}^{-1}$ we denote the set $(R_i^{-1})_{i \in I}$. By $Fd(\mathcal{B})$ we denote the union of $Fd(R_i)$ for all $i \in I$ and call it the field of $\mathcal{B}$.

2.2 Some notions from universal algebra

For A a nonempty set and n a nonnegative integer we define $A^0 = \{\varnothing\}$, and, for $n > 0$, A^n is the set of n-tuples of elements from A. An *n-ary operation* (or *function*) on A is any function f from A^n to A; n is the *arity* (or *rank*) of f. A *finitary* operation is an n-ary operation, for some n. The image of $(a_1, \dots, a_n)$ under an n-ary operation f is denoted by $f(a_1, \dots, a_n)$. An operation f on A is called a *nullary* operation (or *constant*) if its arity is zero; it is completely determined by the image $f(\varnothing)$ in A of the only element $\varnothing$ in A^0, and as such it is convenient to identify it with the element $f(\varnothing)$. Thus a nullary operation is thought of as an element of A. An operation f on A is *unary*, *binary*, or *ternary* if its arity is 1, 2, or 3, respectively.

Definition 1 (Algebraic signature)**.** An *algebraic signature* (also *type*) is a set

$\mathcal{F}$ of *function symbols* such that a nonnegative integer n is assigned to each member f of $\mathcal{F}$. This integer is called the *arity* (or *rank*) of f, and f is said to be an n-ary function symbol. The subset of n-ary function symbols in $\mathcal{F}$ is denoted by $\mathcal{F}_n$.

Definition 2 (Algebra)**.** If $\mathcal{F}$ is a language of algebras then an *algebra* $\mathbf{A}$ of *type* $\mathcal{F}$ is an ordered pair (A, F) where A is a nonempty set and F is a family of finitary operations on A indexed by the language F such that corresponding to each n-ary function symbol f in F there is an n-ary operation $f^{\mathbf{A}}$ on A. The set A is called the *underlying set* of $\mathbf{A} = (A, F)$, and the $f^{\mathbf{A}}$'s are called the *fundamental operations* of $\mathbf{A}$. (In practice we prefer to write just f for $f^{\mathbf{A}}$.) If $\mathcal{F}$ is finite, say $\mathcal{F} = \{f_1, \ldots, f_k\}$, we often write $(A, f_1, \ldots, f_k)$ for (A, F).

An algebra $\mathbf{A}$ is *finite* if the cardinality of its underlying set, denoted as $|A|$, is finite. Following the model-theoretic convention, we may write $x \in \mathbf{A}$, meaning that x belongs to the underlying set of $\mathbf{A}$.

Definition 3 (Homomorphism)**.** Suppose $\mathbf{A}$ and $\mathbf{B}$ are two algebras of the same type $\mathcal{F}$. A mapping $\alpha : A \to B$ is called a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ if

$$\alpha f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(\alpha a_1, \ldots, \alpha a_n)$$

for each n-ary f in $\mathcal{F}$ and each sequence $a_1, \ldots, a_n$ from A. An *isomorphism* is a homomorphism which is bijective (both injective and surjective). In case $\mathbf{A} = \mathbf{B}$, a homomorphism is also called an *endomorphism* and an isomorphism is referred to as an *automorphism*.

$\mathbf{A}$ is said to be *isomorphic* to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$, if there is an isomorphism from $\mathbf{A}$ to $\mathbf{B}$.

Definition 4 (Subalgebra)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same type. Then $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$ and every fundamental operation of $\mathbf{B}$ is the restriction of the corresponding operation of $\mathbf{A}$, i.e., for each function symbol f, $f^{\mathbf{B}}$ is $f^{\mathbf{A}}$ restricted to B; we write simply $\mathbf{B} \leq \mathbf{A}$.

A *subuniverse* of $\mathbf{A}$ is a subset B of A which is closed under the fundamental operations of $\mathbf{A}$, i.e., if f is a fundamental n-ary operation of $\mathbf{A}$ and $a_1, \ldots, a_n \in B$ we would require $f(a_1, \ldots, a_n) \in B$.

Definition 5 (Variety)**.** A nonempty class K of algebras of type $\mathcal{F}$ is called a *variety* if it is closed under subalgebras, homomorphic images, and direct products.

Definition 6 (Equational class)**.** A class K of algebras is an *equational class* if there is a set of equational identities Σ such that K is the class of models of Σ.

In this case we say that K is defined, or axiomatized, by Σ.

Fact 1 (Birkhoff)**.** *K is an equational class iff K is a variety.*

2.3 From lattices to Boolean algebras

A *poset* (partially ordered set) $(L, \leq)$ is called a *lattice* if *supremum* (the least upper bound) and *infimum* (the greatest lower bound) of $\{a, b\}$ exist for all $a, b \in L$. A lattice can be equivalently defined as an algebra.

Definition 7 (Lattice)**.** An algebra $(L, \wedge, \vee)$ with binary operations $\wedge$ (*meet*) and $\vee$ (*join*) is called a *lattice* if:

i) (L1) Idempotency: $a \wedge a = a$, $a \vee a = a$,

ii) (L2) Commutativity: $a \wedge b = b \wedge a$, $a \vee b = b \vee a$,

iii) (L3) Associativity: $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$.

iv) (L4) Absorption identities: $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$.

A lattice L is called *complete* if supremum and infimum exist for any subset $H \subseteq L$. A lattice is called *distributive* if: *i)* $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$, *ii)* $(a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$. A bounded poset is one that has the least element (denoted 0) and the greatest element (denoted 1). In a bounded lattice, a is said to be a *complement* of b iff $a \wedge b = 0$ and $a \vee b = 1$. A *complemented lattice* is a bounded lattice in which every element has a complement. A *Boolean lattice* is a complemented distributive lattice. Thus, in a Boolean lattice, every element a has a unique complement a'.

A *Boolean algebra* is a Boolean lattice in which 0, 1, and $'$ (complementation) are also considered to be fundamental operations. In a Boolean algebra, $\wedge$, 0 and 1 can be defined in terms of $\vee$ and $'$:

$$a \wedge b = (a' \vee b')', \qquad 0 = (a \vee a')', \qquad 1 = a \vee a'.$$

We use the following compact definition of a Boolean algebra:

Definition 8 (Boolean algebra)**.** An algebra $\mathbb{B} = (B, \vee, ')$ with a binary operation $\vee$ (*join*) and a unary operation $'$ (*complement*) is called a Boolean algebra if:

i) (B1) Commutativity: $a \vee b = b \vee a$,

ii) (B2) Associativity: $(a \vee b) \vee c = a \vee (b \vee c)$,

iii) (B3) Huntington's identity: $(a' \vee b')' \vee (a' \vee b)' = a$.

The *Boolean ordering* $\leq$ on $\mathbb{B}$ is defined by $a \leq b$ iff $a \vee b = b$. An *atom* a of $\mathbb{B}$ is a non-zero element that is $\leq$-minimal. A Boolean algebra $\mathbb{B}$ is atomic if for every non-zero element b there exists an atom a such that $a \leq b$. The set of all atoms of $\mathbb{B}$ is denoted by $At(\mathbb{B})$.

Every Boolean algebra is isomorphic to a field of sets. Every complete atomic Boolean algebra is isomorphic to the powerset of some set X, denoted as $\wp(X)$.

2.4 From Boolean algebras to relation algebras

Definition 9. A *Boolean algebra with operators* (Jónsson and Tarski, 1951) is an algebra

$$(A, +, -, f_0, f_1, \dots, f_n),$$

such that $(A, +, -)$ is a Boolean algebra and the functions f_i are *additive* in each argument, i.e., for every f_i with arity k_i, the equality

$$f_i(a + b) = f_i(a) + f_i(b)$$

holds for all $a = (a_1, \dots, a_{k_i}), b = (b_1, \dots, b_{k_i}) \in A^{k_i}$, such that $a_j = b_j$ in all but one index j.

A function $f : A^m \to A$ is said to be *completely additive*, if the existence of $\Sigma_{i\in I} X_i$, where $X_i \in A^m$, implies that $\Sigma_{i\in I} f(X_i)$ also exists and $f(\Sigma_{i\in I} X_i) = \Sigma_{i\in I} f(X_i)$.

A *relation-type algebra* is an algebra

$$\mathbb{A} = (A, +, \cdot, -, 0, 1, ;, \breve{}, 1'), \tag{2.1}$$

with binary operations + (Boolean sum), · (Boolean product) and ; (composition, or relative product), unary operations − (complement) and $\breve{}$ (converse), and constants $0, 1, 1' \in A$ called zero, unit and identity respectively.

Definition 10. A relation-type algebra $\mathbb{A}$ is called a *non-associative algebra* (NA) (Maddux, 1982), if

1) the reduct $(A, +, \cdot, -, 0, 1)$ is a Boolean algebra,

2) $1'; x = x; 1' = x$,

3) Peircean law: $(x; y) \cdot z = 0 \Leftrightarrow (x\breve{}; z) \cdot y = 0 \Leftrightarrow x \cdot (z; y\breve{}) = 0$, for all $x, y, z \in A$.

A non-associative algebra $\mathbb{A}$ is called

a) a *weakly-associative algebra* (WA), if $(1' \cdot x); (1; 1) = ((1' \cdot x); 1); 1$,

b) a *semi-associative algebra* (SA), if $x; (1; 1) = (x; 1); 1$,

c) a *relation algebra* (RA), if $(x; y); z = x; (y; z)$,

for all $x, y, z \in A$.

As usual, $x \leq y$ is used as an abbreviation for $x + y = y$. A non-associative algebra is called *atomic*, if its Boolean reduct is atomic. By $At(\mathbb{A})$ we denote the set of atoms of $\mathbb{A}$. By $At(x)$, where $x \in A$, we denote those atoms a, for which $a \leq x$, e.g., $At(1')$ is the set of identity atoms.

Any complete atomic (particularly any finite) non-associative algebra $\mathbb{A}$ is fully specified by its *atom structure*. An atom structure consists of the set of atoms $At(\mathbb{A})$, the set of identity atoms $At(1') \subseteq At(\mathbb{A})$, the converse restricted to atoms $\breve{}: At(\mathbb{A}) \to At(\mathbb{A})$ and the *composition table*. A composition table is a function $CT : At(\mathbb{A}) \times At(\mathbb{A}) \to \wp(At(\mathbb{A}))$, defined by $z \in CT(x,y)$ iff $(x;y) \cdot z \neq 0$. The triples (x,y,z), where x,y and z are atoms and which satisfy $(x;y) \cdot z \neq 0$ are called *consistent triples*.

A non-associative algebra is said to be *integral* if the composition of any non-zero elements is non-zero. If $\mathbb{A} \in \mathsf{NA}$ and $1' \in At(\mathbb{A})$ then $\mathbb{A}$ is integral. The converse holds if $\mathbb{A} \in \mathsf{SA}$. However, it fails for some $A \in \mathsf{WA}$.

A weakly-associative algebra can be induced on an arbitrary complete Boolean algebra by means of a so-called "notion of consistency".

Definition 11 (Notion of consistency)**.** Let $\mathbb{B} = (B, +, \cdot, -, 0, 1)$ be a Boolean algebra. A *notion of consistency* (Hodkinson, 1997) for $\mathbb{B}$ is a triple $(id, \breve{}, \mathcal{T})$, where $id \in B$, $\breve{}: B \to B$ and $\mathcal{T} \subseteq B^3$, such that

1) $\breve{}$ is a Boolean algebra automorphism on $\mathbb{B}$,

2) $\breve{}$ preserves the id element: $id\breve{} = id$,

3) $\breve{}$ is involutive: $(a\breve{})\breve{} = a$ for every $a \in B$,

for every $a, b, c \in B$:

4) $(a, b, id) \in \mathcal{T} \Leftrightarrow a \cdot b\breve{} = 0$,

5) $(a, b, c) \in \mathcal{T} \Rightarrow (b, c, a) \in \mathcal{T}$ and $(c\breve{}, b\breve{}, a\breve{}) \in \mathcal{T}$,

6) if $b = \Sigma B_0$, where $B_0 \subseteq B$, then
$(a, b, c) \in \mathcal{T} \Leftrightarrow \{(a, b_0, c) \ : \ b_0 \in B_0\} \subseteq \mathcal{T}$.

The elements of $\mathcal{T}$ are called *inconsistent triples* (also *inconsistent triangles*).

Proposition 1 (Hodkinson, 1997)**.** *Assume $\mathbb{B} = (B, +, \cdot, -, 0, 1)$ is a complete Boolean algebra, and $(id, \breve{}, \mathcal{T})$ is a notion of consistency for $\mathbb{B}$. Then the algebra $\mathbb{A} = (B, +, \cdot, -, 0, 1, ;, \breve{}, id)$, where ; is defined as*

$$a; b = -\Sigma\{c \in B \ : \ (a, b, c\breve{}) \in \mathcal{T}\},$$

is a non-associative algebra. If $id \in At(\mathbb{B})$, then $\mathbb{A}$ is a weakly-associative algebra.

2.5 Schröder categories

A *binary relation between sets X and Y* is a triple (X, Y, R), where $R \subseteq X \times Y$. To emphasize that X and Y can be different sets, we call such binary relations *heterogeneous*. If $X = Y$, then we call R a *homogeneous* relation.

Given a heterogeneous binary relation (X, Y, R), one can consider R as a homogeneous binary relation over $X \cup Y$. Conversely, given a homogeneous binary relation R over some set U, one can "convert" it into a heterogeneous binary relation between its domain and codomain.

Some important properties of heterogeneous binary relations are captured by the notion of Schröder categories. Schröder categories are to heterogeneous binary relations what relation algebras are to homogeneous ones.

Definition 12 (Category)**.** A *(small) category* $\mathbb{C}$ (MacLane, 1971) is an algebraic structure consisting of

1) a set of objects $\mathbf{Ob}_{\mathbb{C}}$,

2) a set of arrows $\mathbf{Ar}_{\mathbb{C}}$,

3) two function $dom, cod : \mathbf{Ar}_{\mathbb{C}} \to \mathbf{Ob}_{\mathbb{C}}$ called domain and codomain ($A \xrightarrow{x} B$ is a shortcut for $dom(x) = A$ and $cod(x) = B$),

4) a function $id : \mathbf{Ob}_{\mathbb{C}} \to \mathbf{Ar}_{\mathbb{C}}$, which, for each object A, specifies an arrow id_A called the identity of A,

5) and a partial binary operation $*$ on $\mathbf{Ar}_{\mathbb{C}}$ called composition,

such that

i) $x * y$ is defined iff $cod(x) = dom(y)$ and then $dom(x * y) = dom(x)$, $cod(x * y) = cod(y)$,

ii) associativity: $x * (y * z) = (x * y) * z$ for all $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{z} D$,

iii) unit law: $id_A * x = x = x * id_B$ for all $A \xrightarrow{x} B$.

The set of all arrows of a category $\mathbb{C}$ with a domain A and codomain B is denoted as $\mathbb{C}_{AB}$ and is called a *hom-set*: $\mathbb{C}_{AB} = \{x \in \mathbf{Ar}_{\mathbb{C}} \mid dom(x) = A \text{ and } cod(x) = B\}$. A category has an underlying structure of a directed graph, with vertices $\mathbf{Ob}_{\mathbb{C}}$, arrows $\mathbf{Ar}_{\mathbb{C}}$, source and target functions *dom* and *cod* respectively.

A *covariant (contravariant) functor* $F : \mathbb{C} \to \mathbb{D}$ between categories $\mathbb{C}$ and $\mathbb{D}$ consists of two homonym functions $F : \mathbf{Ob}_{\mathbb{C}} \to \mathbf{Ob}_{\mathbb{D}}$ and $F : \mathbf{Ar}_{\mathbb{C}} \to \mathbf{Ar}_{\mathbb{D}}$, such that

i) covar.: $F(A) \xrightarrow{F(x)} F(B)$ for all $A \xrightarrow{x} B$, contr.: $F(B) \xrightarrow{F(x)} F(A)$ for all $A \xrightarrow{x} B$,

ii) $F(id_A) = id_{F(A)}$,

iii) covar.: $F(x * y) = F(x) * F(y)$, contr.: $F(x * y) = F(y) * F(x)$.

A functor $F : \mathbb{C} \to \mathbb{C}$ is called *involutive*, if, applied twice, it yields the same object or arrow.

Definition 13 (Schröder category)**.** A category $\mathbb{C}$ with partially ordered hom-sets and a contravariant functor $\breve{} : \mathbb{C} \to \mathbb{C}$, which maps $\mathbb{C}_{AB}$ into $\mathbb{C}_{BA}$, is called a Schröder category (Olivier and Serrato, 1980), if

1) each hom-set of $\mathbb{C}$ is a Boolean algebra $(\mathbb{C}_{AB}, 0_{AB}, 1_{AB}, +, \cdot, -)$,

2) $\breve{}$ is involutive,

3) Peircean law: for all $A \xrightarrow{x} B \xrightarrow{y} C$, $A \xrightarrow{z} C$ the following conditions are equivalent: $(x * y) \cdot z = 0_{AC}$, $(x^{\smile} * z) \cdot y = 0_{BC}$ and $(z * y^{\smile}) \cdot x = 0_{AB}$.

A Schröder category with one object is a relation algebra (Jónsson, 1988).

2.6 Connection between Schröder categories and relation algebras

Proposition 2 (Akama, 1998)**.** *Let* $\mathbb{A} = (A,\ 0,\ 1,\ +,\ \cdot,\ -,\ 1',\ \breve{},\ ;)$ *be an atomic non-trivial relation algebra. We define a category* $\mathbb{C}$ *as follows:*

- *Objects:* $\mathbf{Ob}_{\mathbb{C}} = \{x\ :\ x \in A,\ x^{\smile} = x,\ x;x = x\}$
- *Arrows:* $\mathbf{Ar}_{\mathbb{C}} = \{(x, y, z)\ :\ x, z \in \mathbf{Ob}_{\mathbb{C}},\ \ y \in A \text{ and } x;y;z = y\}$
- *Domain and codomain:* $dom(x, y, z) = x$ *and* $cod(x, y, z) = z$.
- *Composition:* $(x, y, z) * (x', y', z')$ *is defined iff* $z = x'$ *and is equal to* $(x, y;y', z')$.
- *Partial order:* $(x, y, z) \leq (x, y', z)$ *iff* $y \leq y'$.
- *Converse:* $(x, y, z)^{\smile} = (z, y^{\smile}, x)$.

where $x, y, z, x', y', z' \in \mathbb{A}$. $\mathbb{C}$ *is a Schröder category, denoted as Split*$(\mathbb{A})$.

Proposition 3 (Jónsson, 1988)**.** *Let* $\mathbb{C}$ *be a Schröder category, and* I *– the set of identity arrows. We define* $Join(\mathbb{C}) = (A,\ 0,\ 1,\ +,\ \cdot,\ -,\ 1',\ \breve{},\ ;)$, *where*

- $(A,\ 0,\ 1,\ +,\ \cdot,\ -)$ *is the direct product of the hom-sets of* $\mathbb{C}$.
- $(a;b)(i,j) := sup\{a(i,k) * b(k,j)\ :\ k \in I\}$ *for* $a, b \in A$ *and* $i, j \in I$.
- $1'(i,j) := \begin{cases} i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.
- $a^{\smile}(i,j) := a(j,i)^{\smile}$.

$Join(\mathbb{C})$ *is a relation algebra.*

2.7 Discussion

In set theory, a binary relation is defined as a set of pairs of related individuals (elements). In logics, the classical calculus of (binary) relations, created by De Morgan, Peirce and Schröder, operates with two kinds of variables: individual variables and relation variables. The theory of relation algebras was intended by Tarski as a theory of binary relations which does not rely on individuals, i.e., in which relations are first-order citizens. This attempt was inspired by strong results of Boole and Stone (Boolean algebras and Stone's representation theorem), who achieved such an abstraction for unary relations.

The study of qualitative calculi relies heavily on the theory of relation algebras. Qualitative calculi are algebraic structures which arise from certain schemes of concrete binary relations. Many qualitative calculi, such as RCC8 (Appendix B.2) or Allen's interval calculus (Appendix B.1), are relation algebras. However, unlike relation algebras, which are an algebraic framework for capturing the *properties* of binary relations, qualitative calculi are focused on *reasoning* with binary relations. For this purpose, even approximations of relational operations, such as composition or converse, may be sufficient.

A relation algebra is a model of certain axioms. Unlike relation algebras, which are defined axiomatically, qualitative calculi are an evolving (and rather applied) framework. New emerging formalisms try to cover the corpus of state-of-the-art algebras of spatio-temporal relations. Thus, qualitative calculi of Ligozat and Renz are non-associative algebras, whereas the more general framework of Dylla et al. admits calculi with weaker algebraic properties that fit into Boolean algebras with operators. An example is the CDR calculus (Appendix B.3).

Categories of relations, like Schröder categories considered in this chapter, or more general structures called *allegories* (Freyd and Scedrov, 1990), are also abstractions of binary relations. Unlike relation algebras, they involve two primitives: objects and arrows (or morphisms). Arrows are abstract binary relations, whereas objects are abstractions of concrete domains and codomains of relations. In this dissertation, I introduce abstract sorts and modularity in qualitative calculi. Modular structures of qualitative calculi, defined in Chapter 6, are related to Schröder categories.

Chapter 3

Qualitative reasoning

Qualitative representation and reasoning is an area of research within knowledge representation and reasoning, which operates with qualitative (as opposed to quantitative) data. It has been studied chiefly in the context of reasonig about time and space. In a nutshell, qualitative (constraint) reasoning deals with the following setting: one has a set of spatial or temporal entities, e.g., regions of a Euclidean space, or time intervals, and a set of asserted relations between these entities, also called constraints. Based on this data one wants to check if these constraints are mutually coherent and to find logical consequences of these constraints. To illustrate this, let us start with a simple example.

Example 1. *Event A is before event B, event C overlaps with event B, event C starts before event A.* Query 1: does *event C* finish after *event A*? Query 2: what is the strongest entailed relation between *event A* and *event C*?

Events in this example occur at time intervals. Formally, a time interval is a pair $A = (A_1, A_2)$ of rational-valued (or real-valued) endpoints. A, B and C are variables ranging over the set of all time intervals. The *universe* of events is the set $\mathcal{U} = \{(x_1, x_2) \ : \ x_1, x_2 \in \mathbb{Q}, \ x_1 < x_2\}$. Consequently, the relations *before*, *overlaps with*, *finishes after* and *starts before* are binary relations over the universe. For example, $starts\_before = \{(A, B) \ : \ A, B \in \mathcal{U}, \ A_x < B_x\}$. We may also use a predicate-style syntax to define a binary relation: $starts\_before(A, B) \equiv_{def} A_x < B_x$.

As formalized above, Example 1 (Query 1) is an instance of the *constraint satisfaction problem* (CSP) (Montanari, 1974). Query 1 can be formulated as follows: given an arbitrary assignment of values from the universe to variables, such that the assignment satisfies all constraints, does the relation *finishes_after* hold between the values of variables C and A? An equivalent task would be to check if there exists a variable assignment such that the constraints are satisfied, along with an additional constraint: *event C* does not finish after *event A*. Indeed, if such valuation exists, then "*event C* finishes after *event A*" is not entailed by the given set of constraints. In CSP, this is called the *satisfiability* task.

To answer Query 2, one has to confine oneself to a (finite) system of predefined relations. In our case it can be the set of temporal relations defined by Allen (1983). Some authors refer to this kind of problems as *CSP with a constraint language* (Bodirsky, 2012, Bodirsky and Dalmau, 2013).

Most CSP solving techniques are applicable to problems with a finite universe. Qualitative reasoning problems that occur in the spatio-temporal domain usually have infinite universes. A way to deal with infinite universes is to use so-called qualitative calculi. A *qualitative calculus* is a structure, in which not the objects, but the relations themselves are first-order citizens. This structure is algebraic, which means that it is characterized solely by functions and distinguished elements, without employing any relation.

The standard reasoning toolboxes for qualitative calculi are SparQ (Wolter, 2009) and GQR (Gantner et al., 2008).

3.1 Abstract partition schemes and their algebras

Let us start by defining *granularity* relations (Cohen-Solal et al., 2015) between two collections[1] of sets. These relations will be used mainly between collections of binary relations.

Definition 14 (Granularity)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be two collections of sets. $\mathcal{X}$ is said to be

- *finer* than $\mathcal{Y}$ if, for every $X \in \mathcal{X}$, there exists $Y \in \mathcal{Y}$ such that $X \subseteq Y$;
- *coarser* than $\mathcal{Y}$ if, for every $X \in \mathcal{X}$, there exists $\mathcal{Y}_0 \subseteq \mathcal{Y}$ such that $X = \cup\mathcal{Y}_0$;
- a *refinement* of $\mathcal{Y}$ if $\mathcal{X}$ is finer than $\mathcal{Y}$ and $\mathcal{Y}$ is coarser than $\mathcal{X}$.

The relations “finer than”, “coarser than” and “refinement of” are transitive.

Definition 15 (Partition)**.** Let X be some nonempty set and $\mathcal{P}$ – a set of its subsets. $\mathcal{P}$ is said to be a *partition* of X if each element of X belongs to one and only one element of $\mathcal{P}$.

The main object of study in this dissertation are partitions of a universal binary relation $U \times U$. Following the terminology of Dylla et al. (2013), we call them abstract partition schemes.

Definition 16 (Abstract partition scheme)**.** A collection $\mathcal{P}$ of binary relations over some set U is called an *abstract partition scheme* over U, if it is a partition of $U \times U$. Then, relations in $\mathcal{B}$ are said to be *jointly exhaustive and pairwise disjoint* (JEPD) on U.

Initially, the notion of a partition scheme was formalized by Ligozat and Renz (2004) in a more restrictive way. In the sequel, I will refer to partition schemes in the sense of Ligozat and Renz as strong partition schemes.

[1] In the sequel, I will use “collection” as a synonym of “set”.

Definition 17 (Strong partition scheme)**.** An abstract partition scheme $\mathcal{P}$ over U is called an *(strong) partition scheme*, if it is closed under converse and contains the identity relation over U.

Let $\mathcal{P}$ be an abstract partition scheme over U. The elements of $\mathcal{P}$ are called $\mathcal{P}$-relations. The set of all unions of $\mathcal{P}$-relations (including the empty union), denoted as $\mathcal{P}_\cup$, is closed under the operations of the Boolean algebra $\wp(U \times U)$, thus it form a subalgebra of the latter. We call $\mathcal{P}_\cup$ the *disjunctive expansion* of $\mathcal{P}$.

The Boolean algebra $\mathcal{P}_\cup$ is complete and atomic: its atoms are the elements of $\mathcal{P}$. The atoms of $\mathcal{P}_\cup$ are usually called *base ($\mathcal{P}_\cup$-)relations*. $\mathcal{P}_\cup$ is isomorphic to the powerset $\wp(\mathcal{P})$, thus each $\mathcal{P}_\cup$-relation is identified by the set of constituting base relations. The elements of $\mathcal{P}_\cup$ are called *(general) $\mathcal{P}_\cup$-relations.* If a $\mathcal{P}_\cup$-relation is a union of two or more base relations, it is said to be *disjunctive*.

Generally speaking, the composition of $\mathcal{P}_\cup$-relations may not be a $\mathcal{P}_\cup$-relation. Moreover, as shown by Scivos and Nebel (2001), a partition scheme may not have a finite refinement closed under composition. The least (in the sense of Boolean inclusion) $\mathcal{P}_\cup$-relation that contains the composition of a pair (R, S) of $\mathcal{P}_\cup$-relations is called their *weak composition* (Düntsch, 2003, Ligozat and Renz, 2004).

Definition 18 (Weak composition)**.** Given a Boolean algebra $\mathcal{P}_\cup$, *weak composition*, denoted by $\diamond$, is a binary operation on $\mathcal{P}_\cup$ defined as

$$R \diamond S = \cup\{T \in \mathcal{P} \ : \ T \cap (R \circ S) \neq \varnothing\}.$$

As opposed to $\diamond$, the usual composition $\circ$ is referred to as *strong* (or *extensional*) composition.

Similarly, the set of $\mathcal{P}$-relation may not be closed under the converse operation. The least $\mathcal{P}$-relation that contains R^{-1} is called the *weak converse* (Dylla et al., 2013) of R, as opposed to the *strong converse* $^{-1}$.

Definition 19 (Weak converse)**.** Given a Boolean algebra $\mathcal{P}_\cup$, *weak converse*, denoted by $^\smile$, is a unary operation on $\mathcal{P}_\cup$ defined as

$$R^\smile = \cup\{S \in \mathcal{P} \ : \ S \cap R^{-1} \neq \varnothing\}.$$

Definition 20 (Algebra generated by an abstract partition scheme)**.** The *algebra*

$$\mathbb{A}_\mathcal{P} = (\mathcal{P}_\cup, \cup, \cap, -_{U\times U}, \varnothing, U \times U, \diamond, ^\smile)$$

is said to be *generated by the abstract partition scheme* $\mathcal{P}$ (or by the set $\mathcal{P}$ of JEPD relations over U).

$\mathbb{A}_\mathcal{P}$ is said to have strong composition (or strong converse) if, for all $R, S \in \mathcal{P}_\cup$, $R \diamond S = R \circ S$ ($R^\smile = R^{-1}$ respectively).

Proposition 4. *$\mathbb{A}_\mathcal{P}$ is a Boolean algebra with operators.*

Proof. The statement means that $\diamond$ and $\breve{}$ are additive in each argument (Definition 9). We prove a stronger statement in Proposition 5. □

Proposition 5. *Let $\mathcal{P}$ be an arbitrary (finite or infinite) abstract partition scheme over a set U. Then weak composition and weak converse operations defined on the disjunctive expansion of $\mathcal{P}$ are completely additive, i.e., completely distribute over the union.*

Proof. The set of binary relations over U, with ordinary relational operations, is a relation algebra, noted $\mathfrak{Re}(U)$. In any relation algebra, composition and converse are completely additive (Chin and Tarski, 1951, Theorems 1.1 and 2.3). Thus, (strong) composition and converse completely distribute over the union in $\mathfrak{Re}(U)$. This means that $(\cup X) \circ (\cup Y) = \cup\{x \circ y \ : \ x \in X \text{ and } y \in Y\}$ and $(\cup X)^{-1} = \cup\{x^{-1} \ : \ x \in X\}$. The existence of $\cup X$ and $\cup Y$ is provided by the fact that $\mathfrak{Re}(U)$ is complete and atomic (because its Boolean reduct is a powerset).

Let us consider two $\mathcal{P}_{\cup}$-relations, $\mathbf{R} = \cup_{i \in I} R_i$ and $\mathbf{S} = \cup_{j \in J} S_j$, where $R_i, S_j \in \mathcal{P}$ for all $i \in I$ and $j \in J$. By definition of weak composition,

$$\mathbf{R} \diamond \mathbf{S} = \cup \{T \in \mathcal{P} \ : \ T \cap [(\cup_{i \in I} R_i) \circ (\cup_{j \in J} S_j)] \neq \varnothing\}.$$

Using the complete additivity of composition, union and intersection, and the property $\cup X \neq \varnothing \Leftrightarrow \exists x \in X$ such that $x \neq \varnothing$, we obtain that

$$\begin{aligned}
&\{T \in \mathcal{P} \ : \ T \cap ((\cup_{i \in I} R_i) \circ (\cup_{j \in J} S_j)) \neq \varnothing\} \\
&= \{T \in \mathcal{P} \ : \ \cup_{i \in I,\ j \in J} (T \cap (R_i \circ S_j)) \neq \varnothing\} \\
&= \cup_{i \in I,\ j \in J} \{T \in \mathcal{P} \ : \ (T \cap (R_i \circ S_j)) \neq \varnothing\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{R} \diamond \mathbf{S} &= \cup \cup_{i \in I,\ j \in J} \{T \in \mathcal{P} \ : \ (T \cap (R_i \circ S_j)) \neq \varnothing\} \\
&= \cup_{i \in I,\ j \in J} \cup \{T \in \mathcal{P} \ : \ (T \cap (R_i \circ S_j)) \neq \varnothing\} \\
&= \cup_{i \in I,\ j \in J} (R_i \diamond S_j).
\end{aligned}$$

The same way we prove that $\mathbf{R}^{\smile} = \cup_{i \in I}(R_i{}^{\smile})$. □

3.2 Constraint languages

A *(relational) constraint language* is given by a collection of relation symbols and their interpretations (Bennett et al., 1997). We use the formal definition of a constraint language as a relational structure in the model-theoretic sense (Hodges, 1993).

Definition 21 (Relational signature)**.** A *relational signature* is a set σ of relation symbols (also called predicate symbols), each with an associated finite arity.

Definition 22 (Relational structure)**.** A *relational structure* over a signature σ, or shortly a σ*-structure*, is a tuple $\Gamma = (\sigma, U, \cdot^{\Gamma})$, where U is a set called the *universe* and $\cdot^{\Gamma}$ is the *interpretation* function defined on σ, which maps each relation symbol with arity n to an n-ary relation over U.

From here on we confine ourselves to binary relations.

We will say that R is a Γ*-relation*, if R is equal to r^{Γ} for some relation symbol $r \in \sigma$. The set of Γ-relations is denoted as σ^{Γ}:

$$\sigma^{\Gamma} = \{r^{\Gamma} \ : \ r \in \sigma\}.$$

We may also write $R \in \Gamma$, meaning that R is a Γ-relation.

When the interpretation of relation symbols in σ is clear from the context, we will specify a constraint language over a finite signature as

$$\Gamma = (U; r_1, r_2, \ldots, r_n),$$

where U is the universe and $r_1, r_2, \ldots, r_n$ are the relation symbols.

Definition 23 (Disjunctive expansion)**.** Let $\Gamma = (\sigma, U, \cdot^{\Gamma})$ be a constraint language. The *disjunctive expansion* of Γ is the constraint language

$$\Gamma_{\vee} = (\widehat{\sigma}, U, \cdot^{\Gamma_{\vee}}),$$

where $\widehat{\sigma}$ consists of all subsets of σ ($\widehat{\sigma} = \wp(\sigma)$) and, for every $\mathbf{r} \in \widehat{\sigma}$, $\mathbf{r}^{\Gamma_{\vee}} = \cup\{r^{\Gamma} \ : \ r \in \mathbf{r}\}$.

The signature of $\Gamma_{\vee}$ can be also defined, following the logical notation, as the set of all disjunctions of relation symbols from σ. For the signature of $\Gamma_{\vee}$ we will use the set-theoretic notation with one reservation: we will identify a singleton set $\{r\} \in \wp(\sigma)$ with the element $r \in \sigma$. Thus, for $r \in \sigma$ we may also write that $r \in \wp(\sigma)$. Also, we may use the relation symbol $\perp$ in the signature of $\Gamma_{\vee}$ to denote the empty relation $\varnothing$.

A σ-structure Γ is said to be an abstract (or strong) partition scheme, if so is the set of binary relations σ^{Γ}. For such Γ, by $\Gamma_{\cup}$ we will denote the disjunctive expansion of σ^{Γ}, i.e., the set $(\sigma^{\Gamma})_{\cup}$, or equivalently, the set $(\widehat{\sigma})^{\Gamma_{\vee}}$:

$$\Gamma_{\cup} = (\sigma^{\Gamma})_{\cup}.$$

We will usually assume that different relation symbols correspond to different relations. In these cases, for a binary relation $R \in \Gamma$, by R^{σ} we will denote the relation symbol $r \in \sigma$, for which $r^{\Gamma} = R$. Assume $\Gamma = (\sigma, U, \cdot^{\Gamma})$ is an abstract partition scheme, and $\Gamma_{\vee} = (\wp(\sigma), U, \cdot^{\Gamma_{\vee}})$ is its disjunctive expansion. If the function $\cdot^{\Gamma} : \sigma \to \sigma^{\Gamma}$ is bijective, then so is the function $\cdot^{\Gamma_{\vee}} : \wp(\sigma) \to \wp(\sigma)^{\Gamma_{\vee}}$. The inverse of $\cdot^{\Gamma_{\vee}}$ is denoted as $\cdot^{\sigma}$. Thus, for $R \in \Gamma_{\vee}$, R^{σ} denotes the set $\{r_i \ : \ i \in I\}$ of relation symbols from σ, for which $\cup\{r_i^{\Gamma} \ : \ i \in I\} = R$.

Assume $\Gamma = (\sigma, U, \cdot^{\Gamma})$ is an abstract partition scheme. Then σ^{Γ} generates an algebra $\mathbb{A}_{\sigma^{\Gamma}}$ with an underlying set $\Gamma_{\cup}$. The function $\cdot^{\sigma} : \Gamma_{\cup} \to \wp(\sigma)$ is bijective. It induces the algebraic structure of $\mathbb{A}_{\sigma^\Gamma}$ on the set $\wp(\sigma)$. We denote the isomorphic image of the algebra $\mathbb{A}_{\sigma^\Gamma}$ as $\mathbb{A}_\Gamma$ and call it the *algebra generated by the constraint language* Γ:

$$\mathbb{A}_\Gamma = (\wp(\sigma), \cup, \cap, -_\sigma, \varnothing, \sigma, \diamond, \breve{\ }).$$

A σ-language Γ is said to be *finer* than, *coarser* than, or a *refinement* of a σ'-language Γ', if the corresponding relation (see Definition 14) holds between the set of Γ-relations and a set of Γ'-relations.

3.3 Constraint-satisfaction problem

The notion of constraint-satisfaction problem (CSP) was introduced by Montanari (1974). We follow the homomorphism-based definition of CSP (Feder and Vardi, 1998).

Definition 24 (Homomorphism between σ-structures)**.** A *homomorphism* h between two binary σ-structures Γ and Γ' with universes U and U' respectively, is a function from U to U', which preserves all Γ-relations, that is, if $(a_1, a_2) \in r^\Gamma$ for some $r \in \sigma$, then $(h(a_1), h(a_2)) \in r^{\Gamma'}$. An *isomorphism* is a bijective homomorphism h such that its inverse $h^{-1} : U' \to U$ is a homomorphism between Γ' and Γ.

Definition 25 (CSP for a constraint language)**.** Let Γ be a (possible infinite) relational structure with a finite signature σ. Then CSP(Γ) is the computational problem to decide whether a given finite σ-structure I homomorphically maps to Γ.

We assume the reader is familiar with some basic CSP notions, such as *constraint network*, *constraint propagation* or *path-consistency*. For a relevant background we refer to Rossi et al. (2006).

3.4 Frameworks for qualitative calculi

Allen's interval calculus introduced the paradigm of algebraic representation and reasoning into the temporal domain. Since 1983, numerous spatio-temporal calculi have been introduced. Systematic study of such calculi calls for formalizing the notion of a *qualitative calculus* (also called a *qualitative constraint calculus*). The notion of a qualitative calculus has been formalized gradually. Below we present the main milestones and give some definitions of qualitative calculi found in the literature.

3.4.1 Ladkin and Maddux

Ladkin and Maddux (1994) introduce *relation algebras* into qualitative reasoning. It was shown that:

1. A representable relation algebra, equipped with a fixed representation, can be used for expressing a binary constraint satisfaction problem.

2. Any relation algebra is a deductive system which admits the path-consistency method.

The interval calculus can be seen as a relation algebra represented over the set $\mathbb{Q} \times \mathbb{Q}$. However, many qualitative calculi do not fit into this framework.

A number of results from the theory of relation algebras, particularly concerning the *network satisfaction problem*, can be applied within this framework (Ladkin and Maddux, 1994, Hirsch, 1997, Hirsch, 2000).

3.4.2 Nebel and Scivos

Nebel and Scivos (2002) pointed out that the main algorithmic method that is used in qualitative calculi is *constraint propagation* in the form of the *path-consistency method*. The applicability of the path-consistency method requires the set of relations which constitute the constraint language be closed under converse, finite intersection and composition. This is captured in the notion of a constraint algebra.

Definition 26 (Constraint algebra)**.** A set of binary relations $\mathcal{B}$ is called a *constraint algebra* if it is closed under converse, finite intersection and composition.

A constraint algebra is a substructure of a *proper relation algebra* (PRA) (Maddux, 1990), where a PRA has to be additionally closed under complement and finite union. The interesting point about proper relation algebras is that they have an axiomatic counterpart, where the properties of the relation system is described purely axiomatically. However, relation algebras are a bit more powerful than is needed for constraint propagation.

There exist constraint languages for which 3-consistency is sufficient to decide satisfiability. These constraint languages may not be closed under union and complement. One may confine oneself with such a sublanguage of a constraint language for the sake of better computational properties. For this reason, one may miss important properties when viewing constraint languages as PRAs.

Conclusions Nebel and Scivos (2002) identified the essential properties of qualitative calculi. However, qualitative calculi were not formally defined. A qualitative calculus is seen as a constraint language which has the structure of a constraint algebra.

3.4.3 Ligozat and Renz

Many qualitative calculi arise from a set of jointly exhaustive and pairwise disjoint (JEPD) relations. The framework of Ligozat and Renz is based on a special class of JEPD relations called partition schemes.

Definition 27 (Qualitative calculus of Ligozat and Renz (QCLR))**.** A *qualitative calculus* is a triple $(\mathbb{A}, U, \varphi)$ such that

- $\mathbb{A} = (A, +, \cdot, -, 0, 1, ;, ^{\smile}, 1')$ is a non-associative algebra,
- $\varphi : \mathbb{A} \to \wp(U \times U)$ is a homomorphism of Boolean algebras,
- $\varphi(1') = Id_U$,
- $\varphi(a^{\smile}) = (\varphi(a))^{-1}$,
- $\varphi(a; b) \supseteq \varphi(a) \circ \varphi(b)$.

The pair (U, φ) is called a *weak representation* of $\mathbb{A}$. A weak representation is said to be *semi-strong*, if

$$\varphi(a; b) = \cap\{\varphi(c) \ : \ \varphi(c) \supseteq \varphi(a) \circ \varphi(b)\}.$$

Conclusions A qualitative calculus is defined as a structure consisting of two components: symbolic and semantic. The symbolic component is a non-associative algebra, the semantic component is a relational system based on a partition scheme. The connection between the two is given by a *weak representation.* In contrast with Ladkin and Maddux, which consider a qualitative calculus as a “relation algebra represented over a set”, Ligozat and Renz define it as a “non-associative algebra weakly represented over a set”.

3.4.4 Westphal, Hué, and Wölfl

Qualitative calculi are based on algebras of relations generated by partition schemes. Each partition scheme can be thought of as a relational structure. Considering first-order sentences over this structure gives a well-defined satisfiability problem without committing to a particular reasoning approach. Thus, one can consider alternatives such as, e.g., first-order theories, rule-based approaches, or structure-specific algorithms.

Westphal et al. (2014) revisit the definition of qualitative calculi and base it on strong partition schemes and the *notion of consistency* (Definition 11). Recall that we denote the disjunctive expansion of a partition scheme $\Gamma = (\sigma, U, \cdot^{\Gamma})$ as $\Gamma_{\vee} = (\wp(\sigma), U, \cdot^{\Gamma_{\vee}})$ (Section 3.2).

Definition 28 (Qualitative calculus of Westphal et al. (QCW))**.** A *qualitative constraint calculus* is a finite weakly-associative algebra

$$\mathbb{A} = \left(\wp(\sigma), \cup, \cap, -, \varnothing, \sigma, ;, ^{\smile}, 1'\right),$$

for which there exists a finite strong partition scheme $\Gamma = (\sigma, U, \cdot^{\Gamma})$, such that $\mathbb{A}$ is the algebra induced on $\wp(\sigma)$ by the notion of consistency $(id, ^{\smile}, \mathcal{T})$ defined as

- $id \in \sigma$ is the relation symbol in σ corresponding to the identity relation over U:
$$id^{\Gamma} = Id_U,$$

- $\breve{}$ is an unary operation on $\wp(\sigma)$ defined by

$$r^{\smile} = s \Leftrightarrow (r^{\Gamma_\vee})^{-1} = s^{\Gamma_\vee},$$

- $\mathcal{T}$ is the set of triples (r, s, t) with $r, s, t \in \wp(\sigma)$, which correspond to inconsistent triples of $\Gamma_\vee$-relations:

$$\mathcal{T} = \big\{(R^\sigma, S^\sigma, T^\sigma) \ : \ R, S, T \in \Gamma_\vee \text{ and } \Gamma_\vee \not\models \exists x, y, z\ (R(x,y) \wedge S(y,z) \wedge T(z,x))\big\}. \quad (3.1)$$

In (Westphal et al., 2014), integral algebras are defined as those which have an atomic identity. According to that definition, any QCW is an integral algebra by construction. However, according to the definition of integrality which we use (page 11) and which is more common in algebra, QCWs are not necessarily integral algebras. In the discussion of this chapter we show that the definition of QCW can be simplified.

3.4.5 Dylla et al.

Some qualitative calculi are not based on strong partition schemes, e.g., the Cardinal Direction (Relations) Calculus (CDR). They may be either not closed under converse or may not contain the identity relation over the universe. The definition of a qualitative calculus given in Dylla et al. (2013) covers calculi arising from an arbitrary finite set of JEPD relations.

Definition 29 (Qualitative calculus of Dylla et al. (QCD))**.** A *qualitative calculus* with binary relations is a tuple $(\mathit{Rel}, \mathit{Int}, \breve{}, \diamond)$ with the following properties:

- *Rel* is a finite, non-empty set of *base relations*. The subsets of *Rel* are called *relations*. We use r, s, t to denote base relations and R, S, T to denote relations.

- $\mathit{Int} = (U, \varphi)$ is an interpretation with a non-empty universe U and a map $\varphi : \mathit{Rel} \to \wp(U \times U)$ with $\{\varphi(r) \mid r \in \mathit{Rel}\}$ being JEPD on U. The map φ is extended to arbitrary relations by setting $\varphi(R) = \bigcup_{r \in R}$ for every $R \subseteq \mathit{Rel}$.

- The *converse operation* $\breve{}$ is a map $\breve{} : \mathit{Rel} \to \wp(\mathit{Rel})$ that satisfies

$$\varphi(r^{\smile}) \supseteq \varphi(r)^{-1}$$

for every $r \in \mathit{Rel}$. The operation $\breve{}$ is extended to arbitrary relations by setting $R^{\smile} = \bigcup_{r \in R} r^{\smile}$ for every $R \subseteq \mathit{Rel}$.

- The *composition operation* $\diamond$ is a map $\diamond : \mathit{Rel} \times \mathit{Rel} \to \wp(\mathit{Rel})$ that satisfies

$$\varphi(r \diamond s) \supseteq \varphi(r) \circ \varphi(s)$$

for all $r, s \in Rel$. The operation $\diamond$ is extended to arbitrary relations by setting $R \diamond S = \bigcup_{r \in R} \bigcup_{s \in S} r \diamond s$ for every $R, S \in Rel$.

Definition 30. Let $C = (Rel, Int, \breve{}, \diamond)$ be a qualitative calculus.
C has *weak converse* if, for all $r \in Rel$:

$$r^{\smile} = \bigcap \{S \subseteq Rel \ : \ \varphi(S) \supseteq \varphi(r)^{\smile}\}. \tag{3.2}$$

C has *strong converse* if, for all $r \in Rel$:

$$\varphi(r^{\smile}) = \varphi(r)^{\smile}. \tag{3.3}$$

C has *weak composition* if, for all $r, s \in Rel$:

$$r \diamond s = \bigcap \{T \subseteq Rel \ : \ \varphi(T) \supseteq \varphi(r) \circ \varphi(s)\}. \tag{3.4}$$

C has *strong composition* if, for all $r \in Rel$:

$$\varphi(r \diamond s) = \varphi(r) \circ \varphi(s). \tag{3.5}$$

R_1	$r \cup s = s \cup r$	$\cup$-commutativity
R_2	$r \cup (s \cup t) = (r \cup s) \cup t$	$\cup$-associativity
R_3	$\overline{\overline{r} \cup \overline{s}} \cup \overline{\overline{r} \cup s} = r$	Huntington's axiom
R_4	$r \diamond (s \diamond t) = (r \diamond s) \cup t$	$\diamond$-associativity
R_5	$(r \cup s) \diamond t = (r \diamond t) \cup (s \diamond t)$	$\diamond$-distributivity
R_6	$r \diamond id = r$	identity law
R_7	$(r^{\smile})^{\smile} = r$	$\smile$-involution
R_8	$(r \cup s)^{\smile} = r^{\smile} \cup s^{\smile}$	$\smile$-distributivity
R_9	$(r \diamond s)^{\smile} = s^{\smile} \diamond r^{\smile}$	$\smile$-involutive distributivity
R_{10}	$r^{\smile} \diamond \overline{r \diamond s} \cup \overline{s} = \overline{s}$	Tarski/de Morgan axiom
WA	$((r \cap id) \diamond 1) \diamond 1 = (r \cap id) \diamond 1$	weak $\diamond$-associativity
SA	$(r \diamond 1) \diamond 1 = r \diamond 1$	$\diamond$ semi-associativity
R_{6l}	$id \diamond r = r$	left-identity law
PL	$(r \diamond s) \cap t^{\smile} = \varnothing \Leftrightarrow (s \diamond t) \cap r^{\smile} = \varnothing$	Peircean law

Table 3.1: Axioms for relation algebras and weaker variants.

Proposition 6 (Dylla et al. 2013)**.** *Every qualitative calculus satisfies* $R1 - R3$*,* $R5$*,* $R_7^{\supseteq}$ *,* $R8$*,* $WA^{\supseteq}$*,* $SA^{\supseteq}$ *for all (base and complex) relations. This axiom set is maximal: each of the remaining axioms in Table 3.1 is not satisfied by some qualitative calculus.*

QCD compared with QCLR A weak representation in the sense of Ligozat and Renz (2004) is a calculus with identity relation, strong converse and abstract composition. QCD is more general than QCLR: it does not require an identity

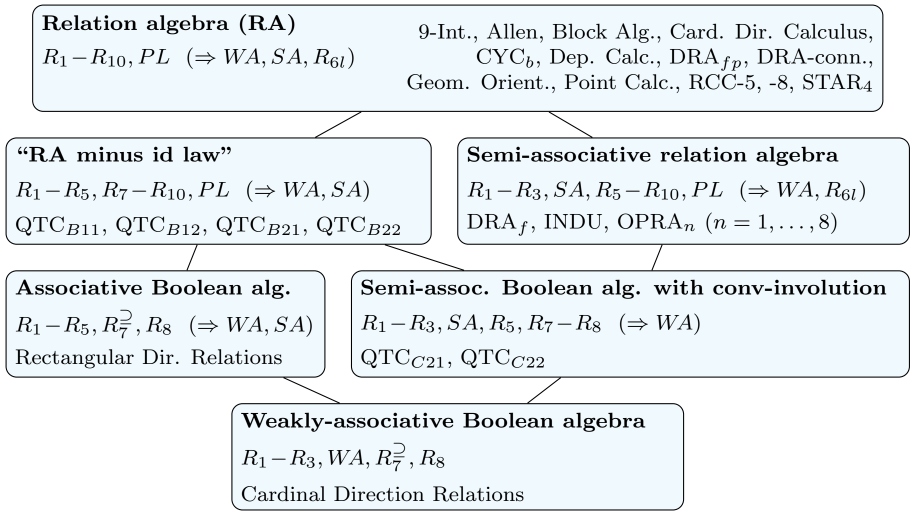


Figure 3.1: Overview of algebra notions and calculi tested (Dylla et al., 2013).

relation, and it only requires abstract converse and composition. Conversely, QCLR is slightly more general than QCD, because the map φ does not need to be injective.

Conclusions The framework of Dylla et al. admits calculi with weak algebraic properties. There is a gap between the formalism and the existing corpus of qualitative calculi. Indeed, all qualitative calculi considered in Dylla et al. (2013) are integral algebras. However, an algebra generated by JEPD relations may not be integral.

3.5 Combination of qualitative calculi

Current methods on combining calculi assume that they are defined over the same universe (Wölfl and Westphal, 2009). Cohn et al. (2014) considers the full combination of RCC8 and RCC8' with the two directional relation models RA and CDC. The *joint satisfaction problem* (JSP) is identified as the main reasoning task. Given a network of topological (RCC8 or RCC8') constraints Θ and a network of directional (RA or CDC) constraints Δ, assuming that Θ and Δ involve the same set of variables, the JSP is to decide when the joint network $\Theta \uplus \Delta$ is satisfiable. $\Theta \uplus \Delta$ is used instead of $\Theta \cup \Delta$ to indicate that Θ and Δ are over the same variables.

3.6 Discussion

Ligozat and Renz (2004) defined a qualitative calculus as "a non-associative algebra with a weak representation". Mossakowski et al. (2006) argued that it is conceptually more adequate to define it as "a non-associative algebra with a semi-strong representation". The first observation is that semi-strong representations are necessarily injective, which is not the case for weak representations. The second observation is that a QCLR $(\mathbb{A}, U, \varphi)$ has (at least) weak composition iff (U, φ) is a semi-strong representation. Otherwise, the composition is said to be *weaker than weak*. By analogy, we will say that a QCD has a semi-strong interpretation (or representation) iff its composition and converse are not weaker than weak. Further, "a QCD with semi-strong interpretation" is the same as "an algebra generated by an abstract partition scheme". Finally, we observe that the definition of QCW can be simplified to "an algebra with identity generated by a partition scheme".

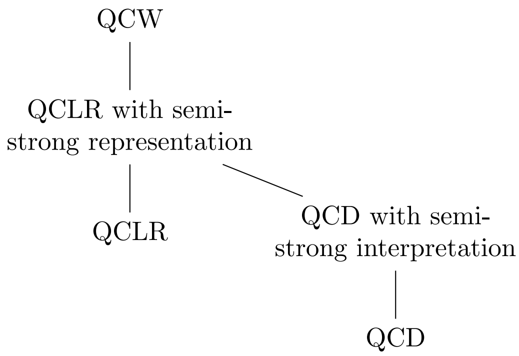


Figure 3.2: A diagram of qualitative calculus frameworks.

3.7 Conclusions

There are various views on what should be called a qualitative calculus. A diagram showing relationships between different frameworks is given in Figure 3.2. In this thesis, I will be considering only those QCLR and QCD which have semi-strong representations. In the sequel, unless stated otherwise, by a "qualitative calculus" I will mean "an algebra generated by an abstract partition scheme". Partition schemes in the sense of Ligozat and Renz (2004) will be referred to as *strong partition schemes*.

Chapter 4

Qualitative reasoning in ontology alignments

Algebras of relations were shown useful in managing ontology alignments. In this chapter, we present the state of the art in algebraic approach to ontology alignments. We discuss the connection between algebras of ontology alignments and qualitative calculi.

The heterogeneity of ontologies on the semantic web requires finding correspondences between them in order to achieve semantic interoperability. The operation of finding correspondences is called ontology matching and its result is a set of correspondences called an alignment (Euzenat and Shvaiko, 2013). Alignments are used for importing data from one ontology to another or for translating queries.

4.1 Logics for ontology alignments

The algebraic approach to reasoning in ontology alignments is an alternative to full logical reasoning. There are various frameworks for distributed ontologies which take the logical approach.

There are several languages that allow for expressing relations across ontologies. Among them are Distributed Description Logics (DDL) (Borgida and Serafini, 2003), $\mathcal{E}$-connections (Kutz et al., 2004), Package-based Description Logics (P-DL) (Bao et al., 2006), Integrated Distributed Description Logics (IDDL) (Zimmermann, 2007) and Distributed Ontology Language (DOL) (Mossakowski et al., 2012). Some other related work about reasoning with ontology alignments can be found in (Zimmermann and Duc, 2008, Le Duc et al., 2010, Le Duc et al., 2013, Codescu et al., 2014).

Mappings between ontologies in DDL assert relations from the perspective of the target ontology. Mappings between concepts are expressed as *bridge rules*, and those between individuals as *individual correspondences*. The key feature of DDL reasoning is subsumption propagation from one ontology to another. Subsumption is not transitive in DDL, thus cannot be propagated by composition. $\mathcal{E}$-connections are a framework for modular ontologies. Connection between ontological modules are established with *links*, which act as inter-ontology properties. A distinctive feature of $\mathcal{E}$-connections is that each ontology module is supposed to model a portion of the domain that is complementary and non-overlapping with respect to the other ontology modules. In that respect, they express relations only across heterogeneous domains. As the domains of ontologies in an $\mathcal{E}$-connection system must be disjoint, it is not possible to have a concept in some ontology module that has subconcepts or instances in another ontology. *Ontology importing*, which is implemented in P-DL, allows for reusing concepts, relations and individuals defined in one ontology inside another ontology. *Alignments* in IDDL constitute a separate layer and can be regarded independently from ontologies. This makes possible to reason about alignments alone, considering them as first class citizens. Some comparative analysis of DDL, $\mathcal{E}$-connections, P-DL and IDDL can be found in (Homola, 2010, Zimmermann, 2013).

An algebraic calculus of alignments is not intended as a proof theory for a particular semantics of alignments. It is a framework, which allows to use custom algebras of relations for inducing operations on alignments.

4.2 Ontology alignments and networks of ontologies

In this section, we give a logical account of ontology alignments and networks of ontologies in the sense of (Euzenat and Shvaiko, 2013).

Definition 31 (Correspondence)**.** Given two ontologies $\mathcal{O}$ and $\mathcal{O}'$, with associated entity languages $Ent(\mathcal{O})$ and $Ent(\mathcal{O}')$, and a set of alignment relations $\mathbf{R}$, a *correspondence* is a triple $(\mathsf{e},\ \mathsf{e'},\ r)$, such that $\mathsf{e} \in Ent(\mathcal{O})$, $\mathsf{e'} \in Ent(\mathcal{O}')$ and $r \in \mathbf{R}$.

A correspondence $(\mathsf{e},\ \mathsf{e'},\ r)$ is an assertion that a certain pragmatic relation denoted by the symbol r holds between the entities e and e'.

The entities can be restricted to a particular kind of terms of the ontology language based on the ontology vocabulary, e.g., named entities. The entity language can also be an extension of the ontology language. For instance, it can be a query language, such as SPARQL (Harris et al., 2013), adding operations for manipulating ontology entities that are not available in the ontology language itself, like concatenating strings or joining relations. The developments of this dissertation are independent of the chosen entity language.

An important component of a correspondence is the relation that holds between the entities. We fix a set of relations $\mathbf{R}$ that is used for expressing the relations between the entities. The set $\mathbf{R}$ can contain relation symbols like $=$, which is used by matching algorithms, or IRIs like `http://www.w3.org/2004/02/skos/extensions#broaderPartitive`. Relations from ontology languages, such as owl:sameAs, owl.differentFrom, owl:equivalentClass, owl:disjointWith, rdfs:subClassOf or rdf:type, can also be used.

An alignment is defined as a set of correspondences.

Definition 32 (Alignment)**.** Given two ontologies $\mathcal{O}$ and $\mathcal{O}'$, an *alignment* is a set of correspondences between pairs of entities belonging to $Ent(\mathcal{O})$ and $Ent(\mathcal{O}')$ respectively.

Definition 33 (Network of ontologies)**.** A *network of ontologies* (Ω, Λ) is made up of a set Ω of ontologies and a set Λ of alignments between these ontologies. We denote by $\Lambda(\mathcal{O}, \mathcal{O}')$ the set of alignments in Λ between $\mathcal{O}$ and $\mathcal{O}'$.

A correspondence is interpreted with respect to three features: a pair of models from each ontology and a semantic structure, denoted as Δ. The class of models of an ontology $\mathcal{O}$ is denoted as $\mathcal{M}(\mathcal{O})$.

Definition 34 (Satisfied correspondence)**.** A correspondence $\mu = (\mathsf{e}, \mathsf{e}', r)$ is satisfied by two models m, m' of $\mathcal{O}, \mathcal{O}'$ for some semantic structure Δ if and only if $(m(\mathsf{e}), m'(\mathsf{e}')) \in r^{\Delta}$, such that r^{Δ} provides the interpretation of the relation r in the structure. This is denoted by $m, m' \models_{\Delta} \mu$.

Three different kinds of semantic structures are outlined in (Zimmermann and Euzenat, 2006): simple, contextualized and integrated. Let us fix two ontologies $\mathcal{O}_1$ and $\mathcal{O}_2$ and their models m_1 and m_2 with domains of interpretation D_1 and D_2 respectively. An integrated semantic structure consists of functions ε_i from the local domains D_i $(i = 1, 2)$ to a global domain D. A simple semantic structure is a particular case of integrated structure: when $D = \cup_i D_i$ and ε_i are the canonical inclusions of D_i into D. Contextualized semantics is given by a family of binary relations r_{ij} $(i = 1, 2)$ between the local domains D_i and D_j.

Below is an example of how relation symbols are interpreted with respect to each semantics. As an example consider the semantics of the relation symbol $\sqsubseteq$ depending on Δ.

$$\begin{aligned}
\sqsubseteq^{simple(\Delta)} &= \{(X, Y) \ : \ X \subseteq D_1, Y \subseteq D_2 \text{ and } X \subseteq Y\} \\
\sqsubseteq^{integrated(\Delta)} &= \{(X, Y) \ : \ X \subseteq D_1, Y \subseteq D_2 \text{ and } \varepsilon_1(X) \subseteq \varepsilon_2(Y)\} \\
\sqsubseteq^{contextual(\Delta)} &= \{(X, Y) \ : \ X \subseteq D_1, Y \subseteq D_2 \text{ and } r_{12}(X) \subseteq Y\}
\end{aligned}$$

If Δ is simple, then $\sqsubseteq^{\Delta}$ depends only on D_1 and D_2. In this case the semantics of $\sqsubseteq$ corresponds to the interpretation of rdfs:subClassOf if we consider $\mathcal{O}_1$ and $\mathcal{O}_2$ as one large ontology. Likewise, the simple semantics of relation symbols $\|$ (disjointness) and $\equiv$ (equivalence) corresponds to owl:disjointWith and owl:equivalentClass.

Definition 35 (Models of alignments)**.** Given two ontologies $\mathcal{O}$ and $\mathcal{O}'$ and an alignment A between these ontologies, a model of this alignment is a triple (m, m', Δ) with $m \in \mathcal{M}(\mathcal{O})$, $m' \in \mathcal{M}(\mathcal{O}')$, and Δ a semantic structure, such that $\forall \mu \in A$, $m, m' \models_{\Delta} \mu$ (denoted by $m, m' \models_{\Delta} A$).

An alignment is said to be *satisfiable* if it has a model. An alignment is said to be *coherent* if, for any of its class entities, it has a model that makes this class non empty.

4.3 Algebraic calculus of ontology alignments

It was shown that algebras of relations are useful for managing ontologies (Euzenat, 2008). The adopted algebraic formalism is Tarskian relation algebras.

Algebras of relations allow for merging alignments conjunctively or disjunctively, amalgamate alignments with relations of different granularity and compose alignments. This may be particularly useful as a fast way to reason about alignments without resorting to full reasoning.

Consider an algebra of relations $\mathbb{A}$. The approach put forward in (Euzenat, 2008) is that we allow any element of $\mathbb{A}$ to be used in a correspondence. In other words, referring to the previous section, we take $\mathbf{R} = \mathbb{A}$.

Each alignment may be normalized through *norm* to contain exactly one correspondence between any two entities. $\mathbb{A}$ induces the following operations on alignments:

$$\mathcal{A} \wedge \mathcal{A}' = norm(\mathcal{A} \cup \mathcal{A}') \tag{4.1}$$

$$\mathcal{A} \vee \mathcal{A}' = \{(\mathsf{e},\ \mathsf{e}',\ r + r') \ :\ (\mathsf{e},\ \mathsf{e}',\ r) \in norm(\mathcal{A}) \wedge (\mathsf{e},\ \mathsf{e}',\ r') \in norm(\mathcal{A}')\} \tag{4.2}$$

$$\mathcal{A}^{\smile} = \{(\mathsf{e}',\ \mathsf{e},\ r^{\smile}) \ :\ (\mathsf{e},\ \mathsf{e}',\ r) \in \mathcal{A}\} \tag{4.3}$$

If there exists an alignment between ontology $\mathcal{O}$ and ontology $\mathcal{O}'$, and another alignment between $\mathcal{O}'$ and a third ontology $\mathcal{O}''$, we would like to find which correspondences hold between $\mathcal{O}$ and $\mathcal{O}''$. The operation that returns this set of correspondences is called composition.

$$\mathcal{A} \circ \mathcal{A}' = norm(\{(\mathsf{e},\ \mathsf{e}'',\ r;s) \ :\ \exists\ (\mathsf{e},\ \mathsf{e}',\ r) \in \mathcal{A} \text{ and } \exists\ (\mathsf{e}',\ \mathsf{e}'',\ s) \in \mathcal{A}'\}) \tag{4.4}$$

We can regard a network of ontologies as a directed graph, with ontologies being vertices and alignments being edges. Moreover, one can assume that there is at most one alignment between any pair of ontologies in the network. A *closure* of a network of ontologies can be computed by applying a path-consistency algorithm, e.g., PC2 (Mackworth and Freuder, 1985), which in essence is an iterative application of

$$\mathcal{A}_{\mathcal{O}_i,\mathcal{O}_j} \leftarrow \mathcal{A}_{\mathcal{O}_i,\mathcal{O}_j} \wedge (\mathcal{A}_{\mathcal{O}_i,\mathcal{O}_k} \circ \mathcal{A}_{\mathcal{O}_k,\mathcal{O}_j}), \tag{4.5}$$

for every triple $(\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k)$ of ontologies in Λ, until a fixed point is reached.

4.4 The algebra $\mathbb{A}5$ of ontology alignment relations

The general approach of algebraic representation and reasoning in ontology alignments was illustrated in (Euzenat, 2008) on a particular algebra $\mathbb{A}5$. It is generated by 5 atoms: $\equiv, \sqsupset, \sqsubset, \between, \|$, which stand for "equivalent to", "more/less general than", "partially overlaps with" and "disjoint with" respectively. The composition table of $\mathbb{A}5$ is given in Table 4.1.

Example 2. For instance, in Figure 4.1, there are two correspondences: "$\mathcal{O}_2$:Serial writer is subsumed by $\mathcal{O}_1$:Successful creator" and "$\mathcal{O}_2$:Serial writer is equivalent to $\mathcal{O}_3$:Popular writer". Subsumption and Equivalence are encoded in $\mathbb{A}5$ as $\{=, \sqsubset\}$ and $\{=\}$ respectively. By composing these relations we infer a correspondence between $\mathcal{O}_1$:Successful creator and $\mathcal{O}_3$:Popular writer:

$$\begin{aligned}
& (\textsf{Successful creator}, \ \textsf{Serial writer}, \ \{\sqsupset, \equiv\}) * (\textsf{Serial writer}, \ \textsf{Popular writer}, \ \{\equiv\}) \\
= \ & (\textsf{Successful creator}, \ \textsf{Popular writer}, \ \{\sqsupset, \equiv\} * \{\equiv\}) \\
= \ & (\textsf{Successful creator}, \ \textsf{Popular writer}, \ (\sqsupset * \equiv) \cup (\equiv * \equiv)) \\
= \ & (\textsf{Successful creator}, \ \textsf{Popular writer}, \ \{\sqsupset\} \cup \{\equiv\}) \\
= \ & (\textsf{Successful creator}, \ \textsf{Popular writer}, \ \{\sqsupset, \equiv\})
\end{aligned}$$

However, the algebra of relations $\mathbb{A}5$ suffers from two problems:

1. $\mathbb{A}5$ covers relations only between classes. This leaves out of scope the relations owl:sameAs (noted $=$), owl:differentFrom (noted $\neq$), which are defined between instances, and the instance-class relation rdf:type ($\in$). Compositional reasoning with these relations may be used for debugging link sets as shown by Example 3.

Example 3. In Figure 4.1, Mystery novelist is disjoint from Academic. However, these classes have instances, between which there exists a same-as link. One would like to compose $\{\ni\} * \{=\} * \{\in\}$ and obtain the relation $\{\equiv, \sqsupset, \sqsubset, \between\}$ between classes. This would be very useful for debugging alignments (or data sets), since the actual relation between Mystery novelist and Academic is $\nparallel$ so the intersection with $\{\|\}$ is empty revealing unsatisfiability. To achieve this one needs to incorporate the instance-class and instance-instance relations $\in, \ni, =$, and the class-class relations of $\mathbb{A}5$, into a single algebra.

To make this work within the considered framework, one needs an algebra incorporating all these relations. This would allow for encoding such RDF triples as correspondences and use them for the refinement and evolution of alignments.

2. The algebraic calculus that $\mathbb{A}5$ induces on alignments does not allow for distinguishing between unsatisfiability and incoherence of alignments. An alignment

$*$	$\equiv$	$\sqsupset$	$\sqsubset$	$\between$	$\|$
$\equiv$	$\equiv$	$\sqsupset$	$\sqsubset$	$\between$	$\|$
$\sqsupset$	$\sqsupset$	$\sqsupset$	$\equiv\sqsupset\sqsubset\between$	$\sqsupset\between$	$\sqsupset\between\|$
$\sqsubset$	$\sqsubset$	$\equiv\sqsupset\sqsubset\between\|$	$\sqsubset$	$\sqsubset\between\|$	$\|$
$\between$	$\between$	$\sqsupset\between\|$	$\sqsubset\between$	$\equiv\sqsupset\sqsubset\between\|$	$\sqsupset\between\|$
$\|$	$\|$	$\|$	$\sqsubset\between\|$	$\sqsubset\between\|$	$\equiv\sqsupset\sqsubset\between\|$

Table 4.1: Composition table of $\mathbb{A}5$.

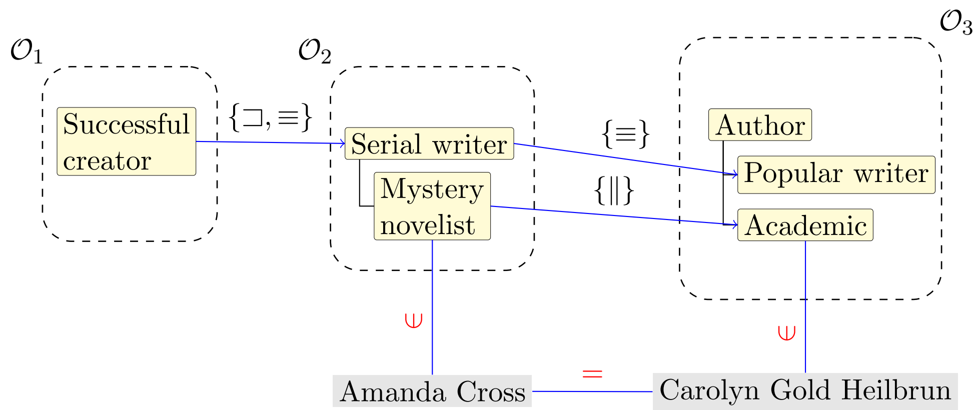


Figure 4.1: An example of unsatisfiability in a linked data sets that can be detected through simple composition of relations across ontologies, data, links and correspondences.

is satisfiable if it has a model, and coherent, if it does not force incoherence on any of its entities. If, by applying algebraic reasoning on alignments, we deduce a correspondence $(\mathsf{C},\ \mathsf{D},\ \varnothing)$, then it means that the alignments are algebraically inconsistent. However, algebraic inconsistency does not imply unsatisfiability, as one would expect. This is illustrated in Example 4.

Example 4. Consider an alignment $\mathcal{A}$ with two correspondences between the same pair of entities: $\mu = (\mathsf{C},\ \mathsf{D},\ \{\|\})$ and $\upsilon = (\mathsf{C},\ \mathsf{D},\ \{\sqsubset, =\})$. Their conjunction is equal to $(\mathsf{C},\ \mathsf{D},\ \varnothing)$:

$$\begin{aligned} \mu \wedge \upsilon &= (\mathsf{C},\ \mathsf{D},\ \{\|\}) \wedge (\mathsf{C},\ \mathsf{D},\ \{\sqsubset, =\}) \\ &= (\mathsf{C},\ \mathsf{D},\ \{\|\} \cap \{=, \sqsubset\}) \\ &= (\mathsf{C},\ \mathsf{D},\ \varnothing)\,. \end{aligned}$$

This means that $\mathcal{A}$ is algebraically inconsistent. But $\mathcal{A}$ has models, thus it is not unsatisfiable. Indeed, if C is interpreted as the empty set, then, whatever the interpretation of D, both μ and υ are satisfied by this interpretation. However, $\mathcal{A}$ is incoherent, since it does not allow the class C to have instances.

A more elaborated example is given in Figure 4.2. The network of ontologies is satisfiable, but the alignment between $\mathcal{O}_1$ and $\mathcal{O}_2$ is incoherent, since it forces the concepts Acceptance and Accepted Paper to have an empty interpretation.

4.5 Discussion

Algebraic reasoning is known to be useful for commonsense reasoning about time and space. This is the inquiry of qualitative spatio-temportal reasoning (QSTR). This chapter has shown that algebraic reasoning may be used also in ontologies,

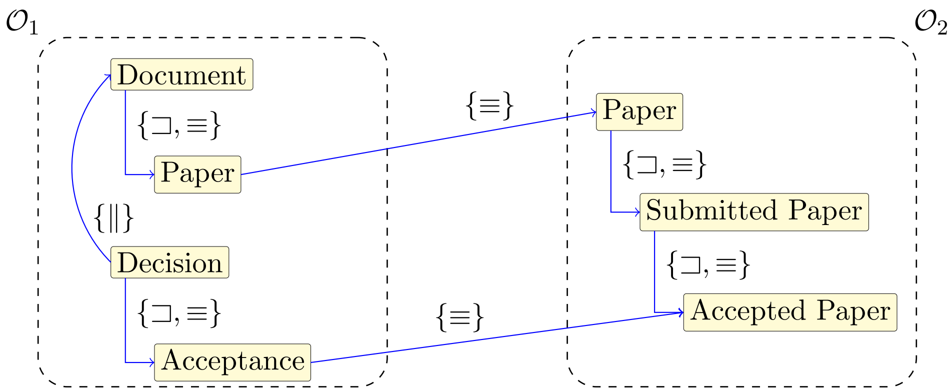


Figure 4.2: An example of a satisfiable but algebraically inconsistent network of ontologies.

particularly for managing ontology alignments. Ontology alignments, and ontologies in general, are a potential area of application of algebraic reasoning techniques, traditionally used in the spatial and temporal domains.

The transition from space and time to arbitrary ontological concepts seems to be a natural generalization. QSTR operates with such concepts as "region", "point", "line", "event", and uses various relations between instances of these concepts, such as the spatial relation "part of" or the temporal relation "during". Like QSTR, ontologies also deal with commonsense knowledge, but are not limited to the spatial or temporal domains. For example, the biomedical ontology SNOMED-CT[1] (Systematized Nomenclature of Medicine – Clinical Terms) contains such concepts as "disease", "medication", "chemical", and relations like "may treat" or "has active ingredient". Like in QSTR, in this example as well some commonsense inference rules may be captured algebraically. For example, the inference rule "if a medication has an active ingredient which may treat a certain disease, then the medication may treat this disease" can be expressed as $may\_treat * has\_active\_ingredient = may\_treat$.

Seeing ontology alignments as a new domain of application of algebraic reasoning techniques, it is evident that the algebraic reasoning approach in ontology alignments may strongly benefit from the theory of qualitative calculi. The relation between algebras of ontology alignment relations and qualitative calculi will be the subject of the current discussion. We will also indicate some particular problems that we will tackle in this thesis based on the theory of qualitative calculi.

Let us start with the problems. The algebra $\mathbb{A}5$ contains relations only between classes. The first problem is how to design an algebra of relations which would incorporate relation between different *kinds* of entities, such as concepts and individuals? The second problem is the following. Having two algebras of relations defined for different kinds of entities, how to combine them into a

[1] https://www.nlm.nih.gov/snomed/

single algebra? For example, one may define an algebra $\mathbb{A}2$ of relations between individuals with two base relations: $=$ and $\neq$. How to combine $\mathbb{A}2$ with $\mathbb{A}5$ into a single algebra? The importance of studying algebraic calculi over heterogeneous universes was justified in Kurata (2009) and some effort to build heterogeneous spatial calculi was made (Kurata and Shi, 2009, Kurata, 2010). However, the state of the art in qualitative reasoning does not offer a general theory for combining algebras of relations for different kinds of entities into a single algebra.

To tackle these problems, it is necessary to take into account the semantics of relation symbols. The connection between the semantics of relations and the algebra of relations is the inquiry of the theory of qualitative calculi. In this chapter, we were talking about algebras of ontology alignment relations and did not use the "qualitative calculus" framework. However, it (the framework) is strongly relevant to the problems that we described.

Let us consider how qualitative calculi can be adapted to ontology alignments. The notion of a qualitative calculus emerged as a formal framework for studying the reasoning properties of numerous algebras of spatial or temporal relations. More formally, as we have seen in Chapter 3, qualitative calculi are a representation and reasoning tool for binary CSP problems with (possibly) infinite universes. A qualitative calculus consists of two components: an abstract algebra of relations and an interpretation structure, which defines the semantics of the relations.

The semantics of alignments is defined in a different way than it is done in qualitative calculi. In qualitative calculi, each relation symbol is interpreted as a binary relation over a fixed universe. We may say that in a qualitative calculus we deal with abstract relations (relation symbols), concrete entities, and an interpretation of relation symbols over the set of entities (the universe). In ontology alignments we deal with *abstract entities* and abstract relations. The semantics of alignments incorporates the semantics of entities (concepts, individuals, properties) and the semantics of relations between entities. We have seen that there are different ways of defining the semantics of alignments.

Consider a network of ontologies (Ω, Λ), such that entities in alignments are terms of the corresponding ontology language. According to the simple semantics of alignments, a model of a network of ontologies (Ω, Λ) is a family $(m_i)_{i \in I}$ of models of $\Omega = (\mathcal{O}_i)_{i \in I}$ with a common domain D. Assume we fix a domain D. Then individuals of Ω are interpreted as elements of D, classes are interpreted as subsets of D, and properties as subsets of $D \times D$.

If we confine ourselves to $\mathbb{A}5$ as a language of ontology alignment relations between classes, then we can talk about concrete interpretations of relation symbols in $\mathbb{A}5$. For example, $\{\sqsubset, \equiv\}$ is interpreted as a binary relation $\{(A, B) \ : \ A, B \in \wp(D) \text{ and } A \subseteq B\}$. The universe of interpretation is the set $\wp(D)$ of all subsets of D.

One assumption that was made in this chapter is that classes that occur in alignments must have nonempty interpretations. This assumption may be natural from the pragmatic point of view, but is not justified from the point of view of simple semantics of alignments, since the latter does not forbid classes to be interpreted as the empty set. However, if we accept this assumption, then the universe of interpretation for the algebra $\mathbb{A}5$ would be the set $U^{(D)} = \wp(D)\setminus\{\varnothing\}$ of all nonempty subsets of D. This yields a qualitative calculus $(\mathbb{A}5, U^{(D)}, \phi)$, in which $\phi : \mathbb{A}5 \to \wp(U \times U)$ is a weak representation. This qualitative calculus is known in spatial reasoning as RCC5 (Jonsson and Drakengren, 1997). A correspondence can be treated as a relational fact between entities. Consider two correspondences (e, e', R_1) and (e', e'', R_2). Composition of R_1 and R_2 infers a relation between e and e'' by eliminating those relations between e and e'' that are logically impossible. If a qualitative calculus has a *weaker than weak* composition, then compositional inference does not eliminate all impossible relations. However, even in that case compositional inference is sound.

Recall that all judgements above assumed a fixed domain of interpretation of the network of ontologies. If compositional inference, defined by the abstract composition operation, is sound for any domain of interpretation D, then we can say that it is sound with respect to the simple semantics of alignments.

So far we discussed the bridge between qualitative calculi and algebras of ontology alignment relations established by model-theoretic semantics of alignments. The model-theoretic semantics of alignments is defined for such relations as subsumption, disjointness, etc. – those relations that are definable in terms of set theory. We call these relations *taxonomical*. A way to go beyond taxonomical relations, i.e., to provide semantics for alignments with non-taxonomical relations, is to interpret relation symbols as axiomatically defined predicates. Thus, relation symbols in $\mathbb{A}5$ may be defined not concretely, i.e., over a fixed universe, but axiomatically. For example, the relation $\{\sqsubset, =\}$ stands for a binary predicate, which is true for any two sets A and B iff $A \subseteq B$. This predicate may be expressed syntactically, as a dyadic formula in an axiomatic set theory, such as ZFC (Zermelo, 1930) or NBG (Von Neumann, 1925). Such syntactic interpretation of $\mathbb{A}5$ would allow for discovering inconsistent triples and building composition tables.

Even though there are methods for building composition tables for axiomatically defined predicates (Randell et al., 1992, Bennett, 1997, Randell and Witkowski, 2002, Wölfl et al., 2007), there is no definition of what is a qualitative calculus with syntactic interpretation. Such a definition should incorporate entites, relations between entities with corresponding relation symbols, and abstract operations on relation symbols. Why do we need such a definition? As it was shown in Chapter 3, a qualitative calculus consists of two components: one symbolic (an algebra of relations) and one semantic (an interpretation structure). Since the symbolic component is derived from the semantic one, to combine the former one has to combine the latter first. Combination of semantic components involves two levels: the combination of entity models and the combination of relational models. Thus, one needs to provide an axiomatic definition not only for relations, but for the universe as well.

4.6 Conclusions

In this chapter, we have considered the algebra $\mathbb{A}5$ of ontology alignment relations and have shown how it can be used for managing ontology alignments. $\mathbb{A}5$ was chosen just as an illustrative example: in effect, the described approach applies to algebras of relations in general. We tackle the problem of combining algebras of ontology alignment relations defined for different kinds of entities. We adopt the "qualitative calculus" framework developed in QSTR. It was shown that the applicability of qualitative calculi is not limited to reasoning about time or space.

The semantics of alignments can be defined either through semantic structures or axiomatically. The first approach allows for using the "qualitative calculus" framework in alignments. For the second approach, in order to benefit from the "qualitative calculus" framework one needs to define what is a syntactic interpretation of a qualitative calculus.

We identified two limitations of the algebra $\mathbb{A}5$ with respect to the simple semantics of alignments. First, $\mathbb{A}5$ covers relations only between classes and does not contain relations with individuals. Second, the calculus that $\mathbb{A}5$ induces on alignments does not allow for distinguishing between unsatisfiability and incoherence of alignments. In Chapter 7, we will introduce a novel qualitative calculus $\mathbb{A}16$, which addresses the limitations of $\mathbb{A}5$.

In addition, we formulated two general problems: how to design a qualitative calculus with relations holding between different kinds of entities, and how to combine qualitative calculi over different universes into a single calculus. These problems are addressed in Chapters 5 and 6, respectively.

Part II

Contribution

Chapter 5

Qualitative calculus of a constraint language

Abstract. Qualitative calculi are generated only by constraint languages based on jointly exhaustive and pairwise disjoint (JEPD) relations. A question arises: how to chose a system of JEPD relations for a given constraint language? The necessary condition is that it must be at least as expressive as the constraint language. Algebraic properties of a qualitative calculus generated by JEPD relations depend on this choice. Here we prove that any constraint language has a qualitative calculus which is a semi-associative algebra. We define the broadest class of partition schemes that generate a weakly-associative algebra. This class contains strong partition schemes of Ligozat and Renz. We fill the gap between the notions "a semi-strong representation of a non-associative algebra" and "an algebra generated by a strong partition scheme".

In Chapter 4, the definition of the qualitative calculus $\mathbb{A}5$ assumes that classes in ontology alignments are interpreted as nonempty sets. However, according to the simple semantics of alignments, classes may be empty. An adjustment into the calculus $\mathbb{A}5$ calls for expanding the $\mathbb{A}5$-relations to the extended universe of classes. This is not possible within the boundaries of strong partition schemes, where the identity is a base relation. In this chapter, I introduce and study partition schemes, in which the identity is a disjunctive relation.

5.1 Qualitative calculus of a constraint language

A constraint language is a natural way of defining a CSP over an infinite domain (Bodirsky and Chen, 2007, Westphal et al., 2014). In qualitative spatio-temporal reasoning, most reasoning problems are given by a language $\Gamma_\vee$, where Γ is jointly exhaustive and pairwise disjoint (JEPD). As it was shown in Section 3.1, JEPD relations induce an algebra called a qualitative calculus.

If a constraint language Γ is coarser than a constraint language Γ' (Section 3.2), then any CSP given in Γ can be given in $\Gamma'_\vee$, i.e., $\Gamma'_\vee$ is more expressive than Γ.

Definition 36 (Calculus of a constraint language)**.** Let Γ be a binary constraint language and $\mathcal{C}$ – a qualitative calculus generated by a JEPD constraint language Γ'. If Γ is coarser than Γ', then we say that $\mathcal{C}$ is a qualitative calculus for Γ, or Γ has a qualitative calculus $\mathcal{C}$.

Consider an example of a constraint language, which is neither jointly exhaustive nor pairwise disjoint.

Example 5. Let X be a nonempty set. Consider the constraint language $\Gamma = (U_X; \subseteq, \supseteq, \|)$, in which $\subseteq$, $\supseteq$ and $\|$ are set-theoretic inclusion and disjointness relations over the universe $U_X = \wp(X)$ consisting of all subsets of X.

Proposition 7. *The constraint language in Example 5 does not have a qualitative calculus in the sense of Westphal et al.*

Proof. Assume Γ is coarser than some partition scheme Γ'. This means that $\subseteq$, $\supseteq$ and $\|$ are $\Gamma'_\vee$-relations. Since $\Gamma'_\vee$-relations are closed under all Boolean operations, all relations obtained from $\subseteq$, $\supseteq$ and $\|$ by applying Boolean operations finitely many times, should be again $\Gamma'_\vee$-relations. Particularly, $\subseteq \cap \supseteq \cap \|$ must be a $\Gamma'_\vee$-relation. But $\subseteq \cap \supseteq \cap \| = \{(\varnothing, \varnothing)\} \subset Id_{U_X}$. This is a contradiction, because Id_{U_X} is a base $\Gamma'_\vee$-relation, and no $\Gamma'_\vee$-relation other than $\varnothing$ can be strictly included in Id_{U_X}. □

Proposition 8. *The constraint language in Example 5 does not have an integral qualitative calculus.*

Proof. In the proof of Proposition 5, it was shown that for any refinement Γ' of Γ, the relations $Id_{U_1} = \subseteq \cap \supseteq \cap \|$ and $Id_{U_2} = \subseteq \cap \supseteq \cap (-\|)$, where $U_1 = \{\varnothing\}$ and $U_2 = U_X \backslash \{\varnothing\}$, are necessarily $\Gamma'_\vee$-relations. From $U_1 \cap U_2 = \varnothing$ follows that $Id_{U_1} \circ Id_{U_2} = \varnothing$, therefore $Id_{U_1} \diamond Id_{U_2} = \varnothing$. Since Id_{U_1} and Id_{U_2} are both nonempty relations and their composition is empty, we conclude that the algebra $\mathbb{A}_{\Gamma'}$ generated by the abstract partition scheme Γ' is nonintegral. □

Partition schemes are an important class of partitions, because the qualitative calculi that they generate satisfy the Pearcean law, the identity law, and as a consequence of these two laws have involutive converse ($r^{\smile\smile} = r$). These algebraic properties have a practical importance. For example, the Pearcean law guarantees that the triangle aRb, bSc, aTc is algebraically consistent iff so is the triangle $bSc, cT^{\smile}a, bR^{\smile}a$. The involutive property of converse is an implicit assumption in most path-consistency algorithms, including PC2. Any qualitative calculus satisfying the Peircean law and the identity law is by definition a non-associative algebra (Section 2.4). A question arises: are partition schemes the broadest class of partitions that generate non-associative algebras? The answer is negative. In the next section we define a broader class of partitions that generate non-associative algebras.

5.2 Non-associative partition schemes

Proposition 7 shows that not every constraint language has a qualitative calculus based on a partition scheme. The problem is that in a partition scheme the identity relation is required to be a base relation. A straightforward solution is to weaken this condition by requiring the identity relation to be a disjunction of base relations. This leads to the following definition.

Definition 37 (Non-associative partition scheme)**.** Let I be a finite index set with a distinguished subset $I_0 \subseteq I$, $\breve{}$ a unary operation on I, U a nonempty set and $(R_i)_{i \in I}$ a family of binary relations on U. The tuple

$$\mathcal{P} = (I, I_0, \breve{}, U, (R_i)_{i \in I}) \tag{5.1}$$

is called a *non-associative partition scheme* if

1) $(R_i)_{i \in I}$ are JEPD on U,

2) $\cup_{i \in I_0} R_i = Id_U$ and

3) $R_{\breve{i}} = R_i^{-1}$ for all $i \in I$.

We will call the elements of $(R_i)_{i \in I_0}$ *base identity relations.*

Proposition 9. *Every finite constraint language is coarser than some non-associative partition scheme.*

Proof. The proof is easily given by construction. Assume Γ is a constraint language on a universe U. Let Γ' be an expansion of Γ obtained by adding the converses of its elements and the identity relation on U. Then the Boolean algebra $\mathbb{B}$ generated by the binary relations of Γ' has as atoms all nonempty intersections of the generators, i.e., the binary relations in Γ', and their complements (see Givant and Halmos, 2009):

$$At(\mathbb{B}) = \{ \bigcap_{R \in \Gamma'} \pm R \neq \varnothing \},$$

where $\pm R$ denotes either R or its complement $(U \times U) \backslash R$.

To prove that $At(\mathbb{B})$ is closed under converse, assume $S \in At(\mathbb{B})$, then there is an instantiation of $\pm$ given by $\eta : \Gamma' \to \{+, -\}$ such that $S = \cap_{R \in \Gamma'} \eta(R)R$.

$$\begin{aligned} S^{-1} &= \left(\bigcap_{R \in \Gamma'} \eta(R)R \right)^{-1} = \bigcap_{R \in \Gamma'} (\eta(R)R)^{-1} \\ &= \bigcap_{R \in \Gamma'} \eta(R)R^{-1} = \bigcap_{R \in \Gamma'} \eta(R^{-1})R \in At(\mathbb{B}). \end{aligned}$$

Atoms of $\mathbb{B}$ form a non-associative partition scheme on U, and Γ is coarser than $At(\mathbb{B})$. □

An important property that non-associative partition schemes "inherit" from conventional partition schemes is that they also generate non-associative algebras.

Proposition 10. *If $\mathcal{P}$ is a non-associative partition scheme, then the qualitative calculus*

$$\mathbb{A}_{\mathcal{P}} = \left(\mathcal{P}_{\cup}, \varnothing, U \times U, Id_U, -_{U\times U}, ^{-1}, \cup, \cap, \diamond\right), \tag{5.2}$$

in which $\diamond$ is the weak composition of $\mathcal{P}$-relations, is a non-associative algebra.

Proof. The proof is very similar to that of Lemma 2 in Ligozat and Renz (2004). The identity law holds due to the equalities $Id_U \diamond R_i = Id_U \circ R_i = R_i = R_i \circ Id_U = R_i \diamond Id_U$. The Peircean law is proven as follows:

$$(R_i \diamond R_j) \cap R_k^{-1} = \varnothing \Leftrightarrow (R_i \circ R_j) \cap R_k^{-1} = \varnothing \Leftrightarrow (R_j \circ R_k) \cap R_i^{-1} = \varnothing$$
$$\Leftrightarrow (R_j \diamond R_k) \cap R_i^{-1} = \varnothing.$$

□

Definition 38 (Non-associative qualitative calculus)**.** A qualitative calculus is said to be non-associative, if it is generated by a non-associative partition scheme.

We denote the class of non-associative qualitative calculi (up to isomorphism) as NAQC. We will also use the abbreviations PSQC, FNA, FWA for the class of qualitative calculi generated by partition schemes (in the sense of Ligozat and Renz), the class of finite non-associative algebras and the class of finite weakly-associative algebras, respectively. It is obvious that PSQC⊂NAQC. Proposition 10 can be stated as NAQC⊆FNA.

It is known that PSQC⊆FWA and FWA⊈PSQC (Westphal, 2014). The former inclusion does not hold for algebras generated by non-associative partition schemes, as shown below.

Proposition 11. *Qualitative calculi generated by non-associative partition schemes may not be weakly associative (NAQC⊈FWA). There exist finite weakly-associative algebras which are not isomorphic to any qualitative calculus generated by a non-associative partition scheme (FWA⊈NAQC).*

Proof. It is easy to build a non-associative partition scheme, such that it generates a non-weakly-associative algebra. Let $U = \{a, b, c\}$ be a set with 3 distinct elements. Define binary relations R, S, T on U the following way: $R = \{(a, a), (b, b)\}$, $S = \{(c, c)\}$, $T = \{(a, b), (a, c), (b, c)\}$. R, S, T and T^{-1} form a non-associative partition scheme on U. However $S \diamond 1 \neq (S \diamond 1) \diamond 1$. (1 denotes the unit relation: $1 = \{R, S, T, T^{-1}\}$.) Indeed, $S \diamond 1 = S \cup T^{-1}$ and $(S \diamond 1) \diamond 1 = (S \diamond 1) \cup (T^{-1} \diamond 1) = 1$. Since S is an identity atom, we conclude that the algebra generated by the considered non-associative partition scheme is not weakly-associative.

The second statement could be proved relying on (Westphal, 2014, Proposition 3.1), which proves a slightly weaker statement (FWA$\not\subseteq$PSQC). However, we provide an independent, algebraic proof.

Consider the weakly-associative algebra $\mathbb{A}$ (Westphal, 2014) specified in Figure 5.1. Assume there exists a non-associative partition scheme $\mathcal{P} = (I, I_0, \breve{}, U, (R_i)_{i\in I})$ such that $\mathbb{A} \cong \mathbb{A}_{\mathcal{P}}$. The base relations corresponding to $id, r, r^{\smile}, s, s^{\smile}$ are $Id_U, R, R^{-1}, S, S^{-1}$ respectively. Let $|U| = n$. Since $R \circ R^{-1} \subseteq R \diamond R^{-1} = Id_U$ and $R^{-1} \circ R \subseteq R^{-1} \diamond R = Id_U$, R is functional and injective, and so is R^{-1}. Therefore, $|Dom(R)| = |Cod(R)| = |R| = |R^{-1}| = |Dom(R^{-1})| = |Cod(R^{-1})|$. Moreover, $|Dom(R)| \leq \left[\frac{n}{2}\right]$, where $[x]$ denotes the integer part of x. Indeed, if, for example, $|Dom(R)| = |Cod(R)| > \left[\frac{n}{2}\right]$, then there exist $x \in U$ such that $x \in Dom(R)$ and $x \in Cod(R)$, hence $R \diamond R \neq \varnothing$, which leads to a contradiction.

The same applies to S. Thus, we obtain

$$n^2 = |U \times U| = |Id_U \cup R \cup R^{-1} \cup S \cup S^{-1}| = n + 2|R| + 2|S| \leq n + 2\left[\frac{n}{2}\right]$$
$$\geq n + 4.$$

From $n+4 \leq n^2 \leq n+2\left[\frac{n}{2}\right]$ follows that $3 \leq n$ and $n \leq 2$, which has no solution. Thus, $\mathbb{A} \notin$ NAQC.

;	id	r	$r^{\smile}$	s	$s^{\smile}$
id	id	r	$r^{\smile}$	s	$s^{\smile}$
r	r	0	id	0	0
$r^{\smile}$	$r^{\smile}$	id	0	0	0
s	s	0	0	0	id
$s^{\smile}$	$s^{\smile}$	0	0	id	0

Figure 5.1: A weakly-associative algebra which does not belong to NAQC.

□

5.3 Weakly-associative partition schemes

In this section, I define the class of weakly-associative partition schemes, i.e., those non-associative partition schemes that generate weakly-associative algebras.

Let us go back to the non-associative partition scheme from the proof of Proposition 11. If we split T into $T' = \{(a, b)\}$ and $T'' = \{(a, c), (b, c)\}$, then the resulting partition scheme with 6 base relations generates a weakly-associative algebra. This observation is generalized by the notion of "weakly-associative partition scheme".

Definition 39 (Weakly-associative partition scheme)**.** A non-associative partition scheme $(I, I_0, \breve{}, U, (R_i)_{i\in I})$ is said to be *weakly-associative*, if, for every $i \in I$, there are $j, k \in I_0$ such that $R_i \subseteq Fd(R_j) \times Fd(R_k)$.

We denote $Fd(R_i)$ as U_i for every $i \in I_0$. It is easy to see that $(U_i)_{i\in I_0}$ form a partition of U.

Weakly-associative partition schemes can be defined in a more intuitive way as follows. Let S be a nonempty finite set of *sorts* and $(U_i)_{i\in S}$ – a family of pairwise disjoint sets called *homogeneous* universe of sort i. For every $i, j \in S$, let $\mathcal{P}_{ij}$ be a partition of $U_i \times U_j$ such that *1)* $\mathcal{P}_{ii}$ is a partition scheme on U_i for every $i \in S$ and *2)* $\mathcal{P}_{ji}^{-1} = \mathcal{P}_{ij}$ for every $i \neq j \in S$. In this chapter, however, we will stick to the first definition.

In a weakly-associative partition scheme, the domain and codomain of each base relation belong, by definition, to some homogeneous universe: $Dom(R_i) \subseteq U_j$ and $Cod(R_i) \subseteq U_k$. Since j and k are unique, we will write $Dom^*(R_i) = U_j$ and $Cod^*(R_i) = U_k$. We define Dom^* and Cod^* on all $\mathcal{P}$-relations as $Dom^*(\cup_i R_i) = \cup_i Dom^*(R_i)$ and $Cod^*(\cup_i R_i) = \cup_i Cod^*(R_i)$. In the example considered before, the homogeneous universes are $U_1 = \{a, b\}$ and $U_2 = \{c\}$. Then $T' \subseteq U_1 \times U_1$ and $T'' \subseteq U_1 \times U_2$, $dom(T' \cup T'') = U_1$ and $cod(T' \cup T'') = U_1 \cup U_2$.

The following proposition shows that weakly-associative partition schemes are indeed those non-associative partition schemes that generate weakly-associative algebras. Let us prove a lemma first.

Lemma 1. *Let $\mathcal{P}$ be a weakly-associative partition scheme. Then, for any $i, j \in I$, $R_i \diamond R_j \subseteq \{R_k \;:\; Dom^*(R_k) = Dom^*(R_i),\; Cod^*(R_k) = Cod^*(R_j)\}$.*

Proof. Follows from

$$R_i \circ R_j \subseteq Dom^*(R_i) \times Cod^*(R_j) = \\ = \cup\{R_k \;:\; Dom^*(R_k) = Dom^*(R_i),\; Cod^*(R_k) = Cod^*(R_j)\}$$

and the definition of $\diamond$. □

Proposition 12. *Let $\mathcal{P} = (I, I_0, \breve{}, U, (R_i)_{i\in I})$ be a non-associative partition scheme. The qualitative calculus $\mathbb{A}_\mathcal{P}$ is a weakly-associative algebra if and only if $\mathcal{P}$ is a weakly-associative partition scheme.*

Proof. Necessity. Assume $\mathbb{A}_\mathcal{P}$ is weakly-associative. According to Theorem 3.5 Maddux, 1982, in every weakly-associative algebra $x; x^\smile \cdot 1', x^\smile; x \cdot 1' \in At(\mathbb{A})$ for all $x \in At(\mathbb{A})$. Applied to $\mathbb{A}_\mathcal{P}$ we obtain that for every $i \in I$ there exist $j, k \in I_0$ such that $(R_i \diamond R_i^{-1}) \cap Id_U = R_j$ and $(R_i^{-1} \diamond R_i) \cap Id_U = R_k$. Assume $(x, y) \in R_i$. Then $(x, x) \in (R_i \diamond R_i^{-1}) \cap Id_U$ and $(y, y) \in (R_i^{-1} \diamond R_i) \cap Id_U$, hence $x \in U_j$ and $y \in U_k$. Therefore, $(x, y) \in U_j \times U_k$, from which follows that $R \subseteq U_j \times U_k$.

Sufficiency. It is sufficient to prove that $R_i \diamond 1 = (R_i \diamond 1) \diamond 1$ for every $i \in I_0$ (1 denotes $U \times U$).

$$R_i \diamond 1 = R_i \diamond \cup_{j\in I}(R_j) = \cup_{j\in I}(R_i \diamond R_j). \tag{5.3}$$

$$R_i \diamond R_j = Id_{U_i} \diamond R_j = \begin{cases} \varnothing, & \text{if } Dom^*(R_j) \neq U_i, \\ R_j, & \text{if } Dom^*(R_j) = U_i. \end{cases} \tag{5.4}$$

From (5.4)

$$R_i \diamond 1 = \cup_{j \in I}(R_i \diamond R_j) = \cup\{R_j \ : Dom^*(R_j) = U_i\}. \tag{5.5}$$

From (5.3) and (5.5)

$$Dom^*(R_i \diamond 1) = Dom^*(\cup\{R_j \ : \ Dom^*(R_j) = U_i\}) = U_i. \tag{5.6}$$

Applying Lemma 1, we obtain

$$(R_i \diamond 1) \diamond 1 \subseteq \cup\{R_k \ : \ Dom^*(R_k) = Dom^*(R_i \diamond 1) = U_i\} = R_i \diamond 1. \tag{5.7}$$

The opposite inclusion $R_i \diamond 1 \subseteq (R_i \diamond 1) \diamond 1$ follows from the property $x \leq x;1$, which holds in every non-associative algebra (Maddux, 1982, Theorem 1.13). □

The results of Propositions 10, 11 and 12 are summarized in Figure 5.2.

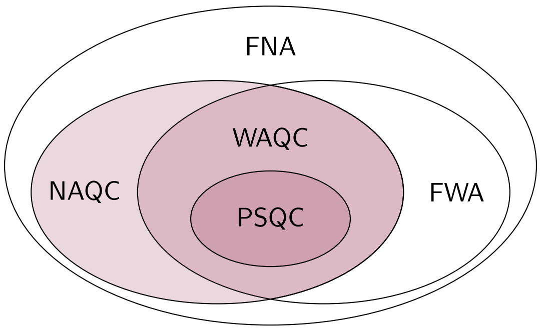


Figure 5.2: The class of non-associative qualitative calculi and its subclasses.

Definition 40 (Weakly-associative qualitative calculus)**.** A qualitative calculus is said to be weakly-associative, if it is generated by a weakly-associative partition scheme.

We denote the class of weakly-associative qualitative calculi as WAQC. Proposition 12 can be stated as WAQC=NAQC∩FWA. The following proposition is stronger than Proposition 9:

Proposition 13. *Every finite constraint language has a weakly-associative qualitative calculus.*

Proof. This follows from Propositin 9, the fact that every non-associative partition scheme can be refined into a weakly-associative partition scheme, and transitivity of refinement. □

A question arises: can the classes NAQC, WAQC or PSQC be axiomatized by equations? The answer to this question is negative, as shown below.

Proposition 14. *If* $\mathbb{A} \in$ *NAQC, then* $\mathbb{A}$ *is directly indecomposable.*

Proof. From $\mathbb{A} \in \mathsf{NAQC}$ follows that there is a non-associative partition scheme $\mathcal{P} = (I, I_0, \breve{}, U, (R_i)_{i \in I})$ such that $\mathbb{A} \cong \mathbb{A}_\mathcal{P}$. For any $R \in \mathbb{A}_\mathcal{P}$ such that $R \neq \varnothing$,

$$(1 \diamond R) \diamond 1 \supseteq (1 \circ R) \circ 1 = 1. \tag{5.8}$$

Assume that $\mathbb{A}_\mathcal{P}$ is directly decomposable, i.e., there exist non-associative partition schemes $\mathcal{P}', \mathcal{P}''$ such that $\mathbb{A}_\mathcal{P} \cong \mathbb{A}_{\mathcal{P}'} \otimes \mathbb{A}_{\mathcal{P}''}$. Let $\mathbb{A} = \mathbb{A}_{\mathcal{P}'} \otimes \mathbb{A}_{\mathcal{P}''}$. Then, for any $R \in \mathbb{A}_{\mathcal{P}'}$, $(R, \varnothing) \in \mathbb{A}$. However,

$$((1,1);(R,\varnothing))\,;(1,1) = ((1 \diamond R) \diamond 1, \varnothing) \neq (1,1). \tag{5.9}$$

Contradiction with (5.8). □

Corollary 1. *NAQC, WAQC and PSQC are not equationally definable.*

Proof. WAQC and PSQC are subclasses of NAQC, thus they also only contain directly indecomposables. According to Birkhoff's theorem (Sankappanavar and Burris, 1981, Theorem 11.9), an equational class should be closed under direct products. □

All atomic and at most countable weakly-associative algebras, unlike non-associative algebras, have relativized representations (Hirsch and Hodkinson, 1997). This propery is inherited by weakly-associative qualitative calculi.

Definition 41 (Semi-associative and associative partition schemes)**.** A weakly-associative partition $\mathcal{P}$ is called *a*) a *semi-associative partition scheme*, if $\mathbb{A}_\mathcal{P}$ is a semi-associative algebra, and *b*) an *associative partition scheme*, if $\mathbb{A}_\mathcal{P}$ is a relation algebra.

The following proposition strengthens the claim of Proposition 13.

Proposition 15. *If every R_i in $\mathcal{P}$ is serial on $Dom^*(R_i)$, then $\mathbb{A}_\mathcal{P}$ is semi-associative. Every binary constraint language has a semi-associative algebra.*

Proof. The first statement is proven by $R_i \diamond (U \times U) = Dom^*(R_i) \times U$ and $(Dom^*(R_i) \times U) \diamond (U \times U) = (Dom^*(R_i) \times U) \circ (U \times U) = Dom^*(R_i) \times U$. The second statement is proven by the fact that every weakly-associative partition scheme can be refined in such a way that every R_i is serial on $Dom^*(R_i)$. □

Let $(\mathbb{A}, \varphi, U)$ be a qualitative calculus, where $\mathbb{A}$ is a non-associative algebra and (φ, U) – its semi-strong representation. Then $\mathcal{P} = \varphi(At(\mathbb{A}))$ is a non-associative partition scheme, and $\mathbb{A} \cong \mathbb{A}_\mathcal{P}$. This shows that "a semi-strong representation of a non-associative algebra" is the same as "an algebra generated by a non-associative partition scheme".

5.4 Discussion

Scivos and Nebel (2004) showed that not every constraint language can be embedded into a finite algebra of binary relations (with strong composition). However, the question whether every constraint language has an associative qualitative calculus (with weak composition) is open. We have shown that every constraint language has at least a semi-associative qualitative calculus.

Combining the results of this chapter with the observations of Section 3.6, we conclude that QCLR and QCD with semi-strong representations, as well as QCW, can be characterized by corresponding classes of partition schemes: non-associative, abstract and strong, respectively. Figure 5.2 shows how the classes of these partition schemes are related.

5.5 Conclusions

Atomicity of the identity relation is required in partition schemes of Ligozat and Renz. However, if we take the framework of qualitative calculi beyond the scope of spatio-temporal reasoning, the necessity of considering partition schemes with disjunctive identity becomes evident. Such partition schemes generate nonintegral algebras.

Motivated by the fact that not every finite constraint language can be embedded into a strong partition scheme (a partition scheme in the sense of Ligozat and Renz), I extended this class of partition scheme. This led to a class of partiton schemes, in which the identity is not a base relation. Such partition schemes generate non-associative algebras, thus I called them non-associative partition schemes. It is known that strong partition schemes generate weakly-associative algebras (Section 3.4.4). This does not hold for non-associative partition schemes. We characterized those non-associative partition schemes that generate weakly-associative algebras. This led to the notion of a weakly-associative partition scheme.

I considered the classes of non-associative and weakly-associative qualitative calculi from the relation- and universal-algebraic point of view and proved that they cannot be axiomatized.

The results of this chapter are used in Chapter 7 to overcome the limitations of the calculus $\mathbb{A}5$ w.r.t. the semantics of alignments. In the next chapter, I generalize the class of weakly-associative partition schemes and introduce modularity in qualitative calculi.

Chapter 6

Modular qualitative calculi

Abstract. In some applications of qualitative reasoning, it is needed to build a large qualitative calculus from smaller modules. The notion of a module can be naturally defined on weakly-associative qualitative calculi. However, weakly-associative partition schemes are not general enough to accommodate relational models such as Cardinal Direction Relations. Here we define a class of modular partition schemes. They are restricted enough to support modularity and loose enough to accept any known qualitative spatio-temporal calculus as a module. We introduce the notion of a modular structure for qualitative calculi generated by modular partition schemes. Modular structure consists of a Boolean lattice of sorts and associates each relation symbol to two sorts by *domain* and *codomain* functions. We define the notion of "relativization to a sort", which allows for extracting a module from a qualitative calculus containing only relations among entities of a given sort.

Kurata and Shi (2009) built a model of cardinal direction relations based on heterogeneous relations which are defined between regions, lines and points. In general, when the universe is heterogeneous, i.e., consists of entities of different kinds, it is desirable to combine qualitative calculi defined only for one kind of entities instead of designing a new qualitative calculus from scratch. This calls for some notion of modularity in qualitative calculi, and a combination operation, which integrates the candidate calculi as modules within a single combined calculus. This chapter introduces the class of modular partition schemes and the corresponding class of modular qualitative calculi. We define the "relativization to a sort" of a modular qualitative calculus and the "combination modulo glue" of two modular qualitative calculi.

Another motivation for defining modularity in qualitative calculi comes from the importance of considering nonintegral qualitative calculi, that is, those calculi in which composition of some nonzero elements is zero. This was justified in the previous chapter, since not all binary constraint languages have an integral qualitative calculus (see Proposition 8). The modular framework which I put forward in this chapter is arguably more adequate for dealing with nonintegral qualitative calculi than the frameworks of Ligozat and Renz or Dylla et al.

In the previous chapter, it was also shown that classes of qualitative calculi, such as the class QCLR, can be characterized by classes of corresponding partition schemes. Thus, QCD, QCLR and QCW correspond to abstract, non-associative and strong partition schemes, respectively. Additionally, we have introduced the class of weakly-associative partition schemes.

The important property of weakly-associative partition schemes is that they allow for abstracting from actual (semantic) domain and codomain functions of binary relations to abstract domain and codomain, defined on the symbolic level. Indeed, in a weakly-associative partition scheme $\mathcal{P}$, each relation $R \in \mathcal{P}$ is required to be a subset of the set $Fd(Id_i) \times Fd(Id_j)$ for some base identity relations $Id_i, Id_j \in \mathcal{P}$. Thus, the domain of R can be associated with Id_i, and codomain with Id_j. This correspondence, established in the semantic level, can be carried over the symbolic level, by associating an atom r with two identity atoms: $(r; r^{\smile}) \cdot 1'$ and $(r^{\smile}; r) \cdot 1'$.

In this chapter, I use the idea of an abstract domain and codomain of relation symbols to introduce modularity in qualitative calculi. This is done by defining an additional symbolic structure, called the modular structure of a qualitative calculus, which maps each relation symbol to abstract sorts, by means of domain and codomain functions. Some restrictions on weakly-associative partition schemes are not necessary to carry out these constructions. Thus, I define a more general class of *modular partition schemes*, which has structural properties that adhere to the notion of modularity. Modular partition schemes are general enough to accommodate relational models such as Cardinal Direction Relations as a module. The class of modular qualitative calculi, that is, the class of algebras generated by modular partition schemes, contains the class of weakly-associative qualitative calculi, but does not contain the class QCLR of non-associative qualitative calculi.

6.1 Modular partition schemes

This section introduces the class of modular partition schemes.

First, let us introduce the class of integral partition schemes, which is a subclass of abstract partition schemes, general enough to accomodate most qualitative spatio-temporal calculi.

Definition 42 (Integral partition scheme)**.** An abstract partition scheme $\mathcal{P}$ is said to be an *integral partition scheme*, if

$$Dom(R_i) \cap Cod(R_j) \neq \varnothing \text{ for any } R_i, R_j \in \mathcal{P}. \tag{6.1}$$

An abstract partition scheme $\mathcal{P}$ over the universe U is said to have a *weak identity relation*, if there exists $Q \in \mathcal{P}$ such that $Q \supseteq Id_U$.

Proposition 16. *An abstract partition scheme generates an integral algebra if and only if it is integral.*

Proof. The algebra $\mathbb{A}_\mathcal{P}$ generated by an abstract partition scheme $\mathcal{P}$ is integral if and only if $R_i \diamond R_j = \varnothing$ for any $R_i, R_j \in \mathcal{P}$. From $R_i \diamond R_j \neq \varnothing \Leftrightarrow R_i \circ R_j \neq \varnothing \Leftrightarrow Cod(R_i) \cap Dom(R_j) \neq \varnothing$ we conclude that $\mathbb{A}_\mathcal{P}$ is integral iff so is the abstract partition scheme $\mathcal{P}$. □

Strong partition schemes may not be integral, and vice versa. All qualitative spatio-temporal calculi considered in (Dylla et al., 2013) are integral, thus they are generated by integral partition schemes. Moreover, all of them are based on a partition scheme with a weak identity. For example, in Cardinal Direction Relations, the relation

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(see Appendix B.3) is a weak identity.

Let $\mathcal{P} = \{R_i \ : \ i \in I\}$ be an abstract partition scheme over some universe U. We denote the $\mathcal{P}_\cup$-relation which is the union of all base relations that have a nonempty intersection with Id_U as $\mathcal{I}$. If $\mathcal{I}$ is not a base relation, then we require that the base relations that constitute $\mathcal{I}$ have disjoint fields:

(*MPS1*) $\mathcal{U} = \{Fd(R_i) \ : \ i \in I, \ R_i \cap Id_U \neq \varnothing\}$ are pairwise disjoint.

Since every $x \in U$ belongs to the field of some base relation in $\mathcal{I}$, the condition (MPS1) ensures that $\mathcal{U}$ makes up a partition of U. Moreover, if $R_i \in \mathcal{I}$, then from (MPS1) follows that $R_i \supseteq Id_{U_0}$, where $U_0 = Fd(R_i)$, and then, $Dom(R_i) = Cod(R_i) = U_0$. This means that R_i is a weak identity over U_0. Finally, based on the premise that $\mathcal{U}$ is a partition of U, we impose an additional requirement:

(*MPS2*) $\{Dom(R_i), Cod(R_i) \ : \ i \in I\}$ is finer than $\mathcal{U}$.

(MPS2) means that for any $R_i \in \mathcal{P}$, there exist $U_j, U_k \in \mathcal{U}$ such that $R_i \subseteq U_j \times U_k$.

Definition 43 (Modular partition scheme)**.** A *modular partition scheme* is an abstract partition scheme that satisfies (MPS1) and (MPS2).

The sets $U_0 \in \mathcal{U}$ are called *homogeneous universes*, whereas the set U is said to be a *heterogeneous universe*. The set $\mathcal{I}$ in a modular partition scheme consists of weak identity relations on the homogeneous universes.

Definition 44 (Strong identity)**.** An abstract partition scheme $\mathcal{P}$ is said to have a *strong identity* if Id_U is a $\mathcal{P}_\cup$-relation.

The term "strong identity" is chosen by analogy with "strong converse" and "strong composition". Thus, non-associative partition schemes have a strong identity and strong converse. If an abstract partition scheme $\mathcal{P}$ has a strong identity, then the condition (MPS1) is automatically satisfied. Weakly-associative partition schemes are modular partition schemes with strong converse and strong identity.

Definition 45 (Strictly-modular partition scheme)**.** An abstract partition scheme is said to be *strictly-modular*, if it satisfies (MPS1), (MPS2) and an additional condition (MPS3):

(*MPS3*) If $U_0 \in \mathcal{U}$, $R_i, R_j \in \mathcal{P}$ and $Cod(R_i), Dom(R_j) \subseteq U_0$, then $Cod(R_i) \cap Dom(R_j) \neq \varnothing$.

6.2 Many-sorted constraint languages

In this section, we introduce many-sorted constraint languages and relate them to modular partition schemes.

Definition 46 (Many-sorted constraint language)**.** A *many-sorted constraint language* is a tuple

$$\Gamma = \left(\tau, \sigma, dom, cod, U, \cdot^{\Gamma}\right),$$

in which

1) τ is a set of unary relation symbols called *sorts*,

2) σ is a set of binary relation symbols called *relations*,

3) $dom, cod : \sigma \to \tau$ are functions from σ to τ, called *(abstract) domain* and *codomain*,

4) $\left(\tau \cup \sigma, U, \cdot^{\Gamma}\right)$ is a relational structure with signature $\tau \cup \sigma$, such that for any $r \in \sigma$,

$$r^{\Gamma} \subseteq dom(r)^{\Gamma} \times cod(r)^{\Gamma}.$$

For any sorts $\mathfrak{s}, \mathfrak{t} \in \tau$, by $\sigma(\mathfrak{s}, \mathfrak{t})$ we denote the set of relation symbols with domain $\mathfrak{s}$ and codomain $\mathfrak{t}$.

$$\sigma(\mathfrak{s}, \mathfrak{t}) = \{r \in \sigma \ : \ dom(r) = \mathfrak{s} \text{ and } cod(r) = \mathfrak{t}\} \tag{6.2}$$

Definition 47 (Disjunctive expansion of a many-sorted constraint language)**.** Let $\Gamma = \left(\tau, \sigma, dom, cod, U, \cdot^{\Gamma}\right)$ be a many-sorted constraint language. The *disjunctive expansion* of Γ is the constraint language

$$\Gamma_{\vee} = \left(\wp(\tau), \wp(\sigma), dom, cod, U, \cdot^{\Gamma_{\vee}}\right),$$

where

1. $dom, cod : \wp(\sigma) \to \wp(\tau)$ are the natural expansions of $dom, cod : \sigma \to \tau$,
2. for every $\mathfrak{s} \in \wp(\tau)$, $\mathfrak{s}^{\Gamma\vee} = \cup\{\mathfrak{s}_0^{\Gamma} \;:\; \mathfrak{s}_0 \in \mathfrak{s}\}$,
3. for every $\mathbf{r} \in \wp(\sigma)$, $\mathbf{r}^{\Gamma\vee} = \cup\{r^{\Gamma} \;:\; r \in \mathbf{r}\}$.

If $\mathfrak{s}, \mathfrak{t} \in \wp(\tau)$, then we define $\sigma(\mathfrak{s}, \mathfrak{t})$ as

$$\sigma(\mathfrak{s}, \mathfrak{t}) = \{\mathbf{r} \in \widehat{\sigma} \;:\; dom(\mathbf{r}) \subseteq \mathfrak{s} \text{ and } cor(\mathbf{r}) \subseteq \mathfrak{t}\} \tag{6.3}$$

Each modular partition scheme (in the sense of Definition 43) can be seen as a many-sorted constraint language, by setting $\tau = \{R_j \;:\; j \in I \text{ and } R_j \cap Id_U \neq \varnothing\}$, $\sigma = \{R_i \;:\; i \in I\}$, and

$$\begin{cases} dom(R_i) = R_j, \\ cod(R_i) = R_k \end{cases} \quad \text{iff} \quad \begin{cases} Dom(R_i) \subseteq Fd(R_j), \\ Cod(R_i) \subseteq Fd(R_k). \end{cases}$$

The conditions (MPS1) and (MPS2) ensure that the definition of functions *dom* and *cod* is correct.

A many-sorted constraint language is said to be a *(strictly-)modular partition scheme*, if the set σ^{Γ} of Γ-relations is a (strictly-)modular partition scheme with homogeneous universes τ^{Γ}. Since, due to (MPS1), modular partition schemes have a weak identity over each local universe, for each sort $\mathfrak{s} \in \tau$ there exists a unique relation $r \in \sigma$ such that $r^{\Gamma} \supseteq Id_{\mathfrak{s}^{\Gamma}}$. We will denote it as $\iota(\mathfrak{s})$, or $\iota_{\mathfrak{s}}$. This defines an injective function

$$\iota : \tau \to \sigma. \tag{6.4}$$

The element $\iota(\tau) = \{\iota_{\mathfrak{s}} \;:\; \mathfrak{s} \in \tau\} \in \wp(\sigma)$ is called the *weak identity element* of the modular partition scheme Γ. Since ι is injective, it is a bijection between τ and $\iota(\tau)$. The disjunctive expansion of $\iota : \tau \to \sigma$ is defined from $\wp(\tau)$ to $\wp(\sigma)$ in a natural way. Thus, ι is a bijection between $\wp(\tau)$ and $\wp(\iota(\tau))$. The inverse of ι is

$$\iota^{-1} : \wp(\iota(\tau)) \to \wp(\tau).$$

6.3 Modular qualitative calculi

A qualitative calculus generated by an abstract partition scheme is a Boolean algebra with two operators (BAO) (see Proposition 4). Since we have distinguished a weak identity element, the algebra generated by a modular partition scheme is a relation-type algebra, i.e., it has the same signature as a relation algebra.

Definition 48 (Modular qualitative calculus)**.** A powerset Boolean algebra $\mathbb{A}$ with operators $\diamond$ and $\breve{}$ and a constant $1'$,

$$\mathbb{A} = \left(\wp(\sigma), \cup, \cap, -, \varnothing, \sigma, \diamond, \breve{}, 1'\right),$$

is said to be a *modular qualitative calculus*, if it is generated by a modular partition scheme, that is, if there exists a modular partition scheme

$$\Gamma = \left(\tau, \sigma, dom, cod, \iota, U, \cdot^{\Gamma}\right),$$

such that $1' = \iota(\tau)$ and $\mathbb{A}$ without the element $1'$ is equal to the algebra $\mathbb{A}_{\Gamma}$ generated by Γ.

We denote the class of modular qualitative calculi (up to isomorphism) as MQC. Figure 6.1 shows how MQC is related to other classes of algebras.

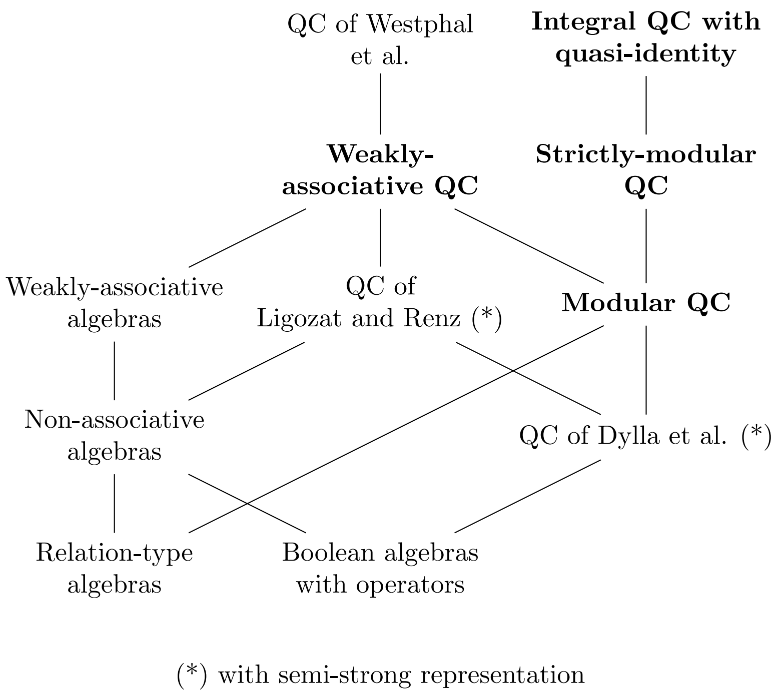


Figure 6.1: Modular qualitative calculi w.r.t. other classes of qualitative calculi and some well-known classes of algebras.

6.4 Modular structure of a qualitative calculus

If a modular qualitative calculus $\mathbb{A}$ is generated by a modular partition scheme $\Gamma = \left(\tau, \sigma, dom, cod, \iota, U, \cdot^{\Gamma}\right)$, then the symbolic structure

$$\mathbb{M} = (\tau, \sigma, dom, cod, \iota)$$

is said to be a *modular structure* of $\mathbb{A}$. We will deliberately use the same notation for the expansions of dom, cod and ι on $\wp(\sigma)$ and $\wp(\tau)$ respectively.

Proposition 17. *If* $\Gamma = \left(\tau, \sigma, dom, cod, \iota, U, \cdot^{\Gamma}\right)$ *is a modular partition scheme and* $\mathbb{A}_\Gamma = (\wp(\sigma), \cup, \cap, -, \varnothing, \sigma, \diamond, ^{\smile}, \iota(\tau))$ *– the algebra generated by* Γ*, then, for any* $r \in \wp(\sigma)$*,*

$$dom(r) = \iota^{-1}\left((r \diamond r^{\smile}) \cap \iota(\tau)\right), \qquad cod(r) = \iota^{-1}\left((r^{\smile} \diamond r) \cap \iota(\tau)\right).$$

Proposition 17 says that, given a modular partition scheme Γ, its domain and codomain functions are uniquely defined by means of the function $\iota^{-1} : \iota(\tau) \to \tau$ and the operations of the algebra $\mathbb{A}_\Gamma$. The proof is given on page 57.

Corollary 2. *Modular structure of a modular qualitative calculus is uniquely defined up to the sort names.*

Proof. This follows from Proposition 17 and the fact that τ is isomorphic to the weak identity element $1'$ of $\mathbb{A}$. □

Corollary 3. *If* $\mathbb{A} = (\wp(\sigma), \cup, \cap, -, \varnothing, \sigma, \diamond, ^{\smile}, 1')$ *is a modular qualitative calculus,* τ *is a set of sorts (unary relation symbols) and* ι *is a bijective function from* τ *to* $1'$*, then* $\mathbb{M} = (\tau, \sigma, dom, cod, \iota)$*, with* $dom, cod : \wp(\sigma) \to \wp(\tau)$ *defined as follows, is a modular structure of* $\mathbb{A}$*.*

$$dom(r) = \iota^{-1}\left((r \diamond r^{\smile}) \cap 1'\right), \qquad cod(r) = \iota^{-1}\left((r^{\smile} \diamond r) \cap 1'\right),$$

where $\iota^{-1} : \wp(1') \to \wp(\tau)$ *is the inverse of* $\iota : \wp(\tau) \to \wp(1')$*.*

6.5 Properties of modular qualitative calculi

Proposition 18. *If* $\mathbb{A}$ *is a modular qualitative calculus and* $\mathbb{M}$ *is its modular structure, then the following properties hold for every* $r, s \in \mathbb{A}$*.*

(MQC1) $dom(r \cup s) = dom(r) \cup dom(s)$,

$cod(r \cup s) = cod(r) \cup cod(s)$,

(MQC2) $r \subseteq 1' \Rightarrow dom(r) = cod(r)$,

(MQC3) $cod(r) \cap dom(s) = \varnothing \Rightarrow r \diamond s = \varnothing$,

(MQC4) $dom(r^{\smile}) = cod(r)$,

$cod(r^{\smile}) = dom(r)$,

(MQC5) $dom(r \diamond s) \subseteq dom(r)$,

$cod(r \diamond s) \subseteq cod(s)$,

(MQC6) $r \subseteq r \diamond 1', \;\; r \subseteq 1' \diamond r$,

(MQC7) $dom(r \diamond 1') = dom(1' \diamond r) = dom(r)$,

$$cod(r \diamond 1') = cod(1' \diamond r) = cod(r),$$

Proof. We assume that $\mathbb{A} = \mathbb{A}_\Gamma$ and $\mathbb{M} = \mathbb{M}_\Gamma$ for some modular partition scheme $\Gamma = \big(\tau, \sigma, dom, cod, \iota, U, \cdot^\Gamma\big)$.

(MQC1). From $dom(r \cup s) = \mathfrak{s}$ follows, by definition of dom and cod, that $Dom((r \cup s)^{\Gamma_\vee}) = \mathfrak{s}^{\Gamma_\vee}$. But $Dom((r \cup s)^{\Gamma_\vee}) = Dom(r^{\Gamma_\vee} \cup s^{\Gamma_\vee}) = Dom(r^{\Gamma_\vee}) \cup Dom(s^{\Gamma_\vee})$, therefore $dom(r) \cup dom(s) = \mathfrak{s}$. The same way we prove that $cod(r \cup s) = cod(r) \cup cod(s)$.

(MQC2). $1' = \iota(\tau)$ by definition of $\mathbb{A}_\Gamma$. $\iota(\tau)$ is the set of weak identity relations on homogeneous universes, therefore, for any subset r of $\iota(\tau)$, $dom(r) = cod(r)$.

(MQC3). $cod(r) \cap dom(s) = \varnothing \Rightarrow Cod(r^{\Gamma_\vee}) \cap Dom(s^{\Gamma_\vee}) = \varnothing \Rightarrow r^{\Gamma_\vee} \circ s^{\Gamma_\vee} = \varnothing \Rightarrow r \diamond s = \varnothing$.

(MQC4). Since $\breve{}$ is additive (Proposition 4), it is enough to prove (MQC4) for $r \in At(\mathbb{A}_\Gamma)$. Assume $dom(r) = \mathfrak{s}$ and $cod(r) = \mathfrak{t}$. Then $r^\Gamma \subseteq \mathfrak{s}^\Gamma \times \mathfrak{t}^\Gamma$, therefore $(r^\Gamma)^{-1} \subseteq \mathfrak{t}^\Gamma \times \mathfrak{s}^\Gamma$. Since $\mathfrak{t}^\Gamma \times \mathfrak{s}^\Gamma$ is a Γ-relation and based on the definition of $(r^\Gamma)^\smile$ as the strongest Γ-relation which approximates $(r^\Gamma)^{-1}$, we obtain $(r^\Gamma)^{-1} \subseteq (r^\Gamma)^\smile \subseteq \mathfrak{t}^\Gamma \times \mathfrak{s}^\Gamma$, therefore $dom(r^\smile) = \mathfrak{t}$ and $cod(r^\smile) = \mathfrak{s}$.

(MQC5). For any $r, s \in At(\mathbb{A}_\Gamma) = \sigma$,

$$r^\Gamma \circ s^\Gamma \subseteq Dom(r^\Gamma) \times Cod(s^\Gamma) \subseteq dom(r)^\Gamma \times cod(s)^\Gamma.$$

Since $dom(r)^\Gamma \times cod(s)^\Gamma$ is a $\Gamma_\vee$-relation, it follows that

$$r^\Gamma \diamond s^\Gamma \subseteq dom(r)^\Gamma \times cod(s)^\Gamma,$$

therefore either $dom(r \diamond s) = cod(r \diamond s) = \varnothing$ or $dom(r \diamond s) = dom(r)$ and $cod(r \diamond s) = cod(s)$. Since dom and cod are additive (MQC1), we obtain that (MQC5) holds for any $r, s \in \mathbb{A}_\Gamma$.

(MQC6) follows from $1'^{\Gamma_\vee} \supseteq Id_U$ and the definition of weak composition.

(MQC7). We will prove that $dom(r \diamond 1') = dom(r)$ for any $r \in \mathbb{A}_\Gamma$. From (MQC6) and (MQC1) it follows that $dom(r \diamond 1') \supseteq dom(r)$, thus it is sufficient to prove that $dom(r \diamond 1') \subseteq dom(r)$, which follows directly from (MQC5).

□

If, for two atoms $r, s \in At(\mathbb{A})$, $cod(r) \neq dom(r)$, then $r \diamond s = \varnothing$. To specify the composition operation, it is enough to do it for the cases when $cod(r) = dom(r)$. Thus, the composition operation of a qualitative calculus is specified by n composition tables, where n is the number of sorts. A modular structure $\mathbb{M}$ can be visualized as a directed graph, in which elements of τ (the sorts) are vertices and each relation symbol $r \in \sigma$ is an arrows from $dom(r)$ to $cod(r)$.

Proposition 19. *If $\mathbb{A}$ is a modular qualitative calculus and $r \in At(\mathbb{A})$, then*

(MQC8) $(r \diamond r^\smile) \cap 1' \in At(\mathbb{A})$ *and* $(r^\smile \diamond r) \cap 1' \in At(\mathbb{A})$

Proof. Let $r \in At(\mathbb{A})$. Since $r \diamond r^{\smile} \neq \varnothing$ and due to (MQC5), $dom(r \diamond r^{\smile}) = dom(r)$. From (MQC5) and (MQC4), $cod(r \diamond r^{\smile}) = cod(r^{\smile}) = dom(r)$. Let $\mathfrak{s} := dom(r)$. We want to prove that $\iota_{\mathfrak{s}} \subseteq r \diamond r^{\smile}$. If $(x, y) \in r^{\Gamma}$, then $(x, x) \in r^{\Gamma} \circ (r^{\Gamma})^{-1} \subseteq r^{\Gamma} \circ (r^{\smile})^{\Gamma} \subseteq r^{\Gamma} \diamond (r^{\smile})^{\Gamma} = (r \diamond r^{\smile})^{\Gamma}$. Also, $(x, x) \in \iota_{\mathfrak{s}}^{\Gamma}$, thus $\iota_{\mathfrak{s}} \cap (r \diamond r^{\smile}) \neq \varnothing$. But since $\iota_{\mathfrak{s}}$ is an atom, we conclude that $\iota_{\mathfrak{s}} \subseteq r \diamond r^{\smile}$. Finally, $(r \diamond r^{\smile}) \cap 1' = \iota_{\mathfrak{s}} \in At(\mathbb{A})$. □

The property (MQC8) is known to hold in all weakly-associative algebras (Maddux, 1982).

Now let us prove Proposition 17.

Proof of Proposition 17. Since ι is bijective, the formula $dom(\mathbf{r}) = \iota^{-1}\left((\mathbf{r} \diamond \mathbf{r}^{\smile}) \cap \iota(\tau)\right)$ is equivalent to $\iota(\mathfrak{s}) = (\mathbf{r} \diamond \mathbf{r}^{\smile}) \cap \iota(\tau)$, where $\mathfrak{s} = dom(\mathbf{r})$. For every $r \in \sigma$, if $r \in \mathbf{r}$, then $r \diamond r^{\smile} \subseteq \mathbf{r} \diamond \mathbf{r}^{\smile}$, due to the additivity of $\diamond$ and $^{\smile}$. From the proof of Proposition 19, $\iota_{dom(r)} \subseteq r \diamond r^{\smile}$, hence $\iota_{dom(r)} \subseteq \mathbf{r} \diamond \mathbf{r}^{\smile}$. Further, $\iota(\mathfrak{s}) = \iota(dom(\mathbf{r})) = \iota(\cup_{r \in \mathbf{r}} dom(r)) = \{\iota_{dom(r)} \ : \ r \in \mathbf{r}\}$. Thus, $\iota(\mathfrak{s}) \subseteq (\mathbf{r} \diamond \mathbf{r}^{\smile})$, and therefore $\iota(\mathfrak{s}) \subseteq (\mathbf{r} \diamond \mathbf{r}^{\smile}) \cap \iota(\tau)$. On the other hand, $dom\left((\mathbf{r} \diamond \mathbf{r}^{\smile}) \cap \iota(\tau)\right) \subseteq dom(\mathbf{r} \diamond \mathbf{r}^{\smile}) \subseteq dom(\mathbf{r}) = \mathfrak{s}$, and similarly $cod\left((\mathbf{r} \diamond \mathbf{r}^{\smile}) \cap \iota(\tau)\right) \subseteq cod(\mathbf{r} \diamond \mathbf{r}^{\smile}) \subseteq cod(\mathbf{r}^{\smile}) = dom(\mathbf{r}) = \mathfrak{s}$. From this we conclude that $(\mathbf{r} \diamond \mathbf{r}^{\smile}) \cap \iota(\tau) \subseteq \iota(\mathfrak{s})$. Thus, $\iota(\mathfrak{s}) = (\mathbf{r} \diamond \mathbf{r}^{\smile}) \cap \iota(\tau)$.

The formula $cod(\mathbf{r}) = \iota^{-1}\left((\mathbf{r}^{\smile} \diamond \mathbf{r}) \cap \iota(\tau)\right)$ is proven in a similar way. □

Proposition 20 (Weak associativity of modular qualitative calculi)**.** *In any modular qualitative calculus* $\mathbb{A} = (\wp(\sigma), \cup, \cap, -, \varnothing, \sigma, \diamond, ^{\smile}, 1')$*,*

(MQC9) $(r \cap 1') \diamond \sigma = ((r \cap 1') \diamond \sigma) \diamond \sigma,$

$$\sigma \diamond (r \cap 1') = \sigma \diamond (\sigma \diamond (r \cap 1')),$$

for any $r \in \mathbb{A}$*.*

Proof. We prove only the first equality, since the second one is proven in a similar way. Assume $\mathbb{A} = \mathbb{A}_{\Gamma}$ for some $\Gamma = \left(\tau, \sigma, dom, cod, \iota, U, \cdot^{\Gamma}\right)$. We need to prove that $\mathbf{r} \diamond \sigma = (\mathbf{r} \diamond \sigma) \diamond \sigma$ for every $\mathbf{r} \subseteq \iota(\tau)$.

For any $\iota_{\mathfrak{s}} \in \mathbf{r}$ and any $t \in \sigma$, if $dom(t) = \mathfrak{s}$, then $t \subseteq \iota_{\mathfrak{s}} \diamond t \subseteq \mathbf{r} \diamond \sigma$. Thus, $\{t \in \sigma \ : \ dom(t) \in dom(\mathbf{r})\} \subseteq \mathbf{r} \diamond \sigma$. Since $dom(\mathbf{r} \diamond \sigma) \subseteq dom(1' \diamond \sigma) = dom(\mathbf{r})$, we conclude that

$$\mathbf{r} \diamond \sigma = \{t \in \sigma \ : \ dom(t) \in dom(\mathbf{r})\}.$$

On the other hand, by (MQC5)

$$dom((\mathbf{r} \diamond \sigma) \diamond \sigma) \subseteq dom(\mathbf{r} \diamond \sigma) \subseteq dom(\mathbf{r}),$$

therefore $(\mathbf{r} \diamond \sigma) \diamond \sigma \subseteq \mathbf{r} \diamond \sigma$.

The opposite inclusion follows from $\mathbf{r} \diamond \sigma \subseteq (\mathbf{r} \diamond \sigma) \diamond 1' \subseteq (\mathbf{r} \diamond \sigma) \diamond \sigma$. □

Proposition 21. *Let* $\mathbb{A}$ *be a modular qualitative calculus with a modular structure* $\mathbb{M}$*. If composition in* $\mathbb{A}$ *is associative and* $1'$ *is its neutral element* ($r \diamond 1' = 1' \diamond r = r$)*, and converse is involutive* ($r^{\smile\smile} = r$)*, then the following defines a Schröder category* $\mathbb{C}$*:*

- *Objects:* $\mathbf{Ob}_{\mathbb{C}} = \wp(\tau)$
- *Arrows:* $\mathbf{Ar}_{\mathbb{C}} = \{(\mathfrak{s}, r, \mathfrak{t}) \ : \ \mathfrak{s}, \mathfrak{t} \in \wp(\sigma) \ \textit{and} \ r \in \sigma(\mathfrak{s}, \mathfrak{t})\}$
- *Identity:* $id_{\mathfrak{s}} = \iota(\mathfrak{s})$,
- *Domain and codomain:* $dom(\mathfrak{s}, r, \mathfrak{t}) = \mathfrak{s}$ *and* $cod(\mathfrak{s}, r, \mathfrak{t}) = \mathfrak{t}$
- *Composition:* $(\mathfrak{s}, r, \mathfrak{t}) * (\mathfrak{t}, r', \mathfrak{u}) = (\mathfrak{s}, r \diamond r', \mathfrak{u})$
- *Partial order:* $(\mathfrak{s}, r, \mathfrak{t}) \leq (\mathfrak{s}, r', \mathfrak{t})$ *iff* $r \subseteq r'$
- *Converse:* $(\mathfrak{s}, r, \mathfrak{t})^{\smile} = (\mathfrak{t}, r^{\smile}, \mathfrak{s})$

Proof. We will first prove that the conditions of the proposition imply that $\mathbb{A}$ is a relation algebra. Let Γ be a modular partition scheme, such that $\mathbb{A} \cong \mathbb{A}_\Gamma$. From the condition $r^{\smile\smile} = r$ follows that Γ has strong converse, i.e., it is closed under $^{-1}$. Since Γ has strong converse, the algebra $\mathbb{A}_\Gamma$ satisfies the Peircean law (see the proof of Proposition 10). Thus, $\mathbb{A}$ also satisfies the Peircean law. Since $\mathbb{A}$ satisfies the identity and the Peircean laws and the composition is associative, it is a relation algebra, by definition.

Let us now prove that $\mathbb{C}$ is a Schröder category. Since there is a one-to-one correspondence between τ and $1'$, we can assume, without loss of generality, that $\tau = 1'$. In Proposition 2, the *Split*$(\cdot)$ operator is defined on atomic relation algebras. The objects of *Split*$(\mathbb{A})$ are those elements $r \in \mathbb{A}$ that satisfy $r^{\smile} = r$ and $r \diamond r = r$. In any relation algebra, if $r \leq 1'$, then $r^{\smile} = r$ and $r \diamond r = r$ (Jónsson and Tarski, 1952). Hence, the subsets of $1'$ are objects in *Split*$(\mathbb{A})$. $\mathbb{C}$ is a subcategory of the category *Split*$(\mathbb{A})$. Moreover, it is a full subcategory, i.e., each hom-set $\mathbb{C}_{st}$ is equal to $Split(\mathbb{A})_{st} = \{(s, r, t) \ : \ s \diamond r \diamond t = r\}$ for all $s, t \subseteq 1'$. This follows from

$$s \diamond r \diamond t = r \ \Leftrightarrow \ dom(r) \subseteq s \text{ and } cod(r) \subseteq t \ \Leftrightarrow r \in \sigma(s, t).$$

□

Objects of $\mathbb{C}$ are sorts, and arrows are relations between these sorts.

6.6 Relativization to a sort

In this section, we define relativization to a sort in modular qualitative calculi.

Definition 49 (Relativization to a sort of a modular partition scheme)**.** Let $\Gamma = \left(\tau, \sigma, dom, cod, \iota, U, \cdot^{\Gamma}\right)$ be a modular partition scheme, and $\mathfrak{s} \subseteq \tau$ – some base or composite sort. The *relativization of* Γ *to the sort* $\mathfrak{s}$ is defined as

$$\Gamma(\mathfrak{s}) = \left(\mathfrak{s}, \sigma_{\mathfrak{s}}, dom, cod, \iota, U_{\mathfrak{s}}, \cdot^{\Gamma}\right),$$

where $\sigma_{\mathfrak{s}} = \sigma(\mathfrak{s}, \mathfrak{s}) = \{r \in \sigma \ : \ dom(r), cod(r) \subseteq \mathfrak{s}\}$ and $U_{\mathfrak{s}} = \mathfrak{s}^{\Gamma}$.

$\Gamma(\mathfrak{s})$ is a modular partition scheme, since it satisfies (MPS1) and (MPS2).

A question arises: how to define relativization on the symbolic structure $(\mathbb{A}, \mathbb{M})$? That is, given a modular qualitative calculus $\mathbb{A}$ with a modular structure $\mathbb{M} = (\tau, \sigma, dom, cod, \iota)$, and a sort $\mathfrak{s} \subseteq \tau$, can we define an algebra $\mathbb{A}(\mathfrak{s})$ in a way that, for any modular partition scheme Γ, if $\mathbb{A} = \mathbb{A}_\Gamma$ and $\mathbb{M} = \mathbb{M}_\Gamma$, then $\mathbb{A}(\mathfrak{s}) = \mathbb{A}_{\Gamma(\mathfrak{s})}$. In other words, we want to define relativization on the symbolic level in a way that it complies with the semantic definition given above.

Definition 50 (Relativization to a sort)**.** Let $\mathbb{A} = (\wp(\sigma), \cup, \cap, -, \varnothing, \sigma, \diamond, \breve{}, 1')$ be a qualitative calculus with a modular structure $\mathbb{M} = (\tau, \sigma, dom, cod, \iota)$, and $\mathfrak{s} \in \wp(\tau)$ be some sort (base or composite). Then the algebra

$$\mathbb{A}(\mathfrak{s}) = (\wp(\sigma_\mathfrak{s}), \cup, \cap, -_\mathfrak{s}, \varnothing, \sigma_\mathfrak{s}, \diamond, \breve{}, \iota(\mathfrak{s}))\,,$$

in which

- $\sigma_\mathfrak{s} = \sigma(\mathfrak{s}, \mathfrak{s}) = \{r \in \sigma \ : \ dom(r), cod(r) \subseteq \mathfrak{s}\}$,
- $-_\mathfrak{s}(r) = (-r) \cap \sigma_\mathfrak{s}$,

is called the *relativization of* $\mathbb{A}$ *to the sort* $\mathfrak{s}$.

The properties (MQC1), (MQC4) and (MQC5) of modular qualitative calculi ensure that $\cup$, $\cap$, $\diamond$ and $\breve{}$ are closed on $\wp(\sigma_\mathfrak{s})$.

Proposition 22. *If* $\mathbb{A} = \mathbb{A}_\Gamma$*, and* $\mathbb{M} = \mathbb{M}_\Gamma$*, then* $\mathbb{A}(\mathfrak{s}) = \mathbb{A}_{\Gamma(\mathfrak{s})}$.

Proof. $\Gamma(\mathfrak{s}) = \left(\mathfrak{s}, \sigma_\mathfrak{s}, dom, cod, \iota, \mathfrak{s}^\Gamma, \cdot^\Gamma\right)$. The algebra generated by $\Gamma(\mathfrak{s})$ is $\mathbb{A}_{\Gamma(\mathfrak{s})} = (\wp(\sigma_\mathfrak{s}), \cup, \cap, -_{\sigma_\mathfrak{s}}, \varnothing, \sigma_\mathfrak{s}, \diamond, \breve{}, \iota(\mathfrak{s}))$. Since $\wp(\sigma_\mathfrak{s}) \subseteq \wp(\sigma)$, the operations $\diamond$ and $\breve{}$ in $\mathbb{A}_{\Gamma(\mathfrak{s})}$ are the restrictions of the corresponding operations of $\mathbb{A}_\Gamma$. By construction, $\mathbb{A}_{\Gamma(\mathfrak{s})}$ is equal to $\mathbb{A}(\mathfrak{s})$. □

Proposition 22 means that the relativization to a sort of a modular qualitative calculus is a modular qualitative calculus. In other words, modular qualitative calculi are closed under relativization to a sort.

6.7 Combination modulo glue

In this section, we will define an operation on modular qualitative calculi called combination modulo glue. As with relativization to a sort, we start with the semantic level.

6.7.1 Operations on modular partition schemes

The *intersection* of two abstract partition schemes $\mathcal{P}_1$ and $\mathcal{P}_2$ over the same universe U is the coarsest refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$, denoted as $Int(\mathcal{P}_1, \mathcal{P}_2)$.

$$Int(\mathcal{P}_1, \mathcal{P}_2) = \{R_i \cap S_j \neq \varnothing \ : \ R_i \in \mathcal{P}_1,\ S_j \in \mathcal{P}_2\}$$

Proposition 23. *Modular partition schemes are closed under intersection, that is, an intersection of two modular partition schemes is again a modular partition scheme. Weakly-associative and strong partition schemes are also closed under intersection.*

Proof. Indeed, an intersection of two weak identity relations is again a weak identity relation, and for any $T = R_i \cap S_i$, such that $R_i \subseteq Fd(Q_1) \times Fd(Q_2)$ and $S_j \subseteq Fd(Q_3) \times Fd(Q_3)$, $T \subseteq Fd(Q_1 \cap Q_3) \times Fd(Q_2 \cap Q_4)$. A modular partition scheme is weakly-associative iff it has strong converse and a strong identity. Intersection of two such partition schemes preserves these properties. A strong partition scheme is a weakly-associative partition scheme, in which the identity is a base relation. Obviously, the intersection of two strong partition scheme is a strong partition scheme. □

Let us define two auxilary operations. Given a set X and a collection of its subsets $\mathcal{X} = \{X_1, \ldots, X_k\}$, the *partition of X induced by $\mathcal{X}$*, denoted as $Part(X, \mathcal{X})$, is defined as the coarsest among those partitions $\mathcal{P}$ of X, for which $\mathcal{X}$ is coarser than $\mathcal{P}$. A constructive definition of $Part(X, \mathcal{X})$ is given in Algorithm 1 (Appendix A). For an abstract partition scheme $\mathcal{P}$ over U, the *grid* of $\mathcal{P}$, noted $Grid(\mathcal{P})$, is defined as:

$$\begin{aligned} Grid(\mathcal{P}) = \{X \times Y \ : \ & X = Fd(Id_U \cap R_i) \neq \varnothing, \\ & Y = Fd(Id_U \cap R_j) \neq \varnothing,\ R_i, R_j \in \mathcal{P}\}. \end{aligned}$$

$Grid(\mathcal{P})$ is an abstract partition scheme over U and satisfies (MPS1) and (MPS2), thus it is a modular partition scheme. The intersection of any abstract partition scheme $\mathcal{P}$ with its grid is the canonical refinement of $\mathcal{P}$ into a modular partition scheme.

Now assume that two modular partition schemes $\mathcal{P}_1$ and $\mathcal{P}_2$ are defined over disjoint universes U_1 and U_2 respectively. Let $R_1, R_2, \ldots, R_n$ be some binary relations between U_1 and U_2.

Definition 51 (Combination of modular partition schemes modulo glue relations)**.** The *combination of $\mathcal{P}_1$ and $\mathcal{P}_2$ modulo $R_1, R_2, \ldots, R_n$*, denoted as $\mathcal{P}_1 \oplus_{R_1,R_2,\ldots,R_n} \mathcal{P}_2$, is defined as the coarsest among partition schemes $\mathcal{P}_3$ over $U_1 \cup U_2$, such that $\mathcal{P}_1$, $\mathcal{P}_2$ and $\{R_1, R_2, \ldots, R_n, R_1^{-1}, R_2^{-1}, \ldots, R_n^{-1}\}$ are coarser than $\mathcal{P}_3$. In this context, the relations $R_1, R_2, \ldots, R_n$ are called *glue relations* between the universes U_1 and U_2.

The constructive definition of combination modulo glue is

$$\mathcal{P}_1 \oplus_{R_1,R_2,\ldots,R_n} \mathcal{P}_2 = Int(\mathcal{P}', \mathcal{P}''), \tag{6.5}$$

where

$$\begin{aligned} \mathcal{P}' &= Grid(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \{U_1 \times U_2,\ U_2 \times U_1\}) \text{ and} \\ \mathcal{P}'' &= \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_{12} \cup \mathcal{P}_{21},\ \text{in which} \\ \mathcal{P}_{12} &= Part(U_1 \times U_2, \{R_1, R_2, \ldots, R_n\}) \text{ and } \mathcal{P}_{21} = \mathcal{P}_{12}^{-1}. \end{aligned}$$

An algorithm which implements (6.5) is given in Algorithm 4 (Appendix A).

6.7.2 A splitted modular qualitative calculus

Here we ask the following question. Let $\Gamma = \left(\tau, \sigma, dom, cod, \iota, U, \cdot^{\Gamma}\right)$ be a modular partition scheme, and let τ_1, τ_2 be some sorts that partition τ, that is, $\tau = \tau_1 \cup \tau_2$ and $\tau_1 \cap \tau_2 = \varnothing$. Assume we are given the relativizations of $\mathbb{A}_\Gamma$ to the sorts τ_1 and τ_2, and the modular structures of these relativizations. What additional information is required to "recover" $\mathbb{A}_\Gamma$ and $\mathbb{M}_\Gamma$ from $\mathbb{A}_\Gamma(\tau_1)$, $\mathbb{M}_\Gamma(\tau_1)$, $\mathbb{A}_\Gamma(\tau_2)$, $\mathbb{M}_\Gamma(\tau_2)$? We will confine ourselves to the restriction that Γ-relations between the sorts τ_1 and τ_2 are "symmetric"[1] with those between τ_1 and τ_2, i.e., if $R \in \Gamma$ and $R \subseteq \tau_1^{\Gamma\vee} \times \tau_2^{\Gamma\vee}$, then $R^{-1} \in \Gamma$.

A modular qualitative calculus is a complete atomic Boolean algebra with completely additive operators (Proposition 5), thus it is completely specified by its *atom structure* $\mathfrak{S}(\mathbb{A})$:

$$\mathfrak{S}(\mathbb{A}) = \left(\sigma, 1', \diamond, \breve{\ }\right), \tag{6.6}$$

where $\diamond : \sigma \times \sigma \to \wp(\sigma)$ and $\breve{\ } : \sigma \to \wp(\sigma)$.

On one hand, we have

$$\mathfrak{S}(\mathbb{A}_\Gamma) = (\sigma, \iota(\tau), \diamond, \breve{\ }), \qquad \mathbb{M}_\Gamma = (\tau, \sigma, dom, cod, \iota). \tag{6.7}$$

On the other hand, we have

$$\begin{aligned} \mathfrak{S}(\mathbb{A}_\Gamma(\tau_1)) &= (\sigma_{\tau_1}, \iota(\tau_1), \diamond, \breve{\ }), & \mathbb{M}_\Gamma(\tau_1) &= (\tau_1, \sigma_{\tau_1}, dom, cod, \iota), \\ \mathfrak{S}(\mathbb{A}_\Gamma(\tau_2)) &= (\sigma_{\tau_2}, \iota(\tau_2), \diamond, \breve{\ }), & \mathbb{M}_\Gamma(\tau_1) &= (\tau_2, \sigma_{\tau_2}, dom, cod, \iota). \end{aligned} \tag{6.8}$$

The "difference" between (6.7) and (6.8) is the following:

$$G_0 = (\sigma_g, dom, cod, \diamond, \breve{\ }), \tag{6.9}$$

where $\sigma_g = \sigma(\tau_1, \tau_2) \cup \sigma(\tau_2, \tau_1)$, $dom, cod : \sigma_g \to \tau$ and

$$\diamond : (\sigma_g \times \sigma) \cup (\sigma \times \sigma_g) \to \wp(\sigma), \qquad \breve{\ } : \sigma_g \to \sigma_g.$$

Since we assumed that $\sigma(\tau_1, \tau_2)$-relations are symmetric with $\sigma(\tau_2, \tau_1)$-relations, we will also use symmetric notation for these relation symbols. Thus, if $r \in \sigma(\tau_1, \tau_2)$, we will denote its converse relation symbol form $\sigma(\tau_2, \tau_1)$ as $r^{\smile}$. We end up with a structure G_1, which is more "compact" than G_0, yet sufficient to reconstruct $\mathbb{A}_\Gamma$ and $\mathbb{M}_\Gamma$ from $\mathbb{A}_\Gamma(\tau_1)$, $\mathbb{M}_\Gamma(\tau_1)$, $\mathbb{A}_\Gamma(\tau_2)$, $\mathbb{M}_\Gamma(\tau_2)$.

$$G_1 = (\sigma(\tau_1, \tau_2), dom, cod, \diamond), \tag{6.10}$$

where $\diamond : (\sigma(\tau_1, \tau_2) \times \sigma) \cup (\sigma \times \sigma(\tau_1, \tau_2)) \to \wp(\sigma)$.

[1] This restriction is not really crucial. However, there are several reasons to impose it. First, to my knowledge there is no justification (so far) of non-symmetric glue's usefullness. Second, this restriction allows for "compressing" the glue. And finally, a generalization which admits non-symmetric glue should be easy to obtain from the symmetric case.

6.7.3 Inconsistent triples

Here we show that in the structure G_1, which "glues together" two splits of a modular qualitative calculus, it is possible to specify the composition by means of inconsistent triples.

Let Γ be a modular partition scheme with a universe U. For $\mathbf{r}, \mathbf{s}, \mathbf{t} \in \wp(\sigma)$, the triple $(\mathbf{r}, \mathbf{s}, \mathbf{t})$ is said to be consistent, if

$$\exists x, y, z \in U \text{ such that } \mathbf{r}^{\Gamma_\vee}(x, y) \wedge \mathbf{s}^{\Gamma_\vee}(y, z) \wedge \mathbf{t}^{\Gamma_\vee}(x, z). \tag{6.11}$$

Otherwise, it is called an *inconsistent triple*. We will denote the set of all inconsistent triples of $\Gamma_\vee$ as $\mathcal{T}$.

Recall that the notion of an inconsistent triple is used in Definition 11. It is slightly different than the one we use: a representation of an inconsistent triple in the sense of Definition 11 satisfies the negation of (6.11), in which $\mathbf{t}^{\Gamma_\vee}(x, z)$ is replaced with $\mathbf{t}^{\Gamma_\vee}(z, x)$.

The following property of inconsistent triples follows straightforwardly from the definition: for any $\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{r}', \mathbf{s}', \mathbf{t}' \in \wp(\sigma)$,

$$\text{If } \mathbf{r}' \subseteq \mathbf{r},\ \mathbf{s}' \subseteq \mathbf{s},\ \mathbf{t}' \subseteq \mathbf{t}, \text{ and } (\mathbf{r}, \mathbf{s}, \mathbf{t}) \in \mathcal{T}, \text{ then } (\mathbf{r}', \mathbf{s}', \mathbf{t}') \in \mathcal{T}. \tag{6.12}$$

From the definition of weak composition on $\Gamma_\vee$ it follows that

$$\mathbf{t} \cap (\mathbf{r} \diamond \mathbf{s}) = \varnothing \quad \Leftrightarrow \quad \mathbf{t}^{\Gamma_\vee} \cap (\mathbf{r}^{\Gamma_\vee} \circ \mathbf{s}^{\Gamma_\vee}) = \varnothing. \tag{6.13}$$

Comparing (6.11) with (6.13) we conclude that $\mathbf{t} \cap (\mathbf{r} \diamond \mathbf{s}) = \varnothing$ iff $(\mathbf{r}, \mathbf{s}, \mathbf{t})$ is an inconsistent triple. Then composition can be defined by means of inconsistent triples as

$$\mathbf{r} \diamond \mathbf{s} = - \cup \{\mathbf{t} \ : \ (\mathbf{r}, \mathbf{s}, \mathbf{t}) \in \mathcal{T}\}. \tag{6.14}$$

Since weak composition is completely additive,

$$\begin{aligned} \mathbf{r} \diamond \mathbf{s} \ &= \cup_{r \in \mathbf{r},\ s \in \mathbf{s}} (- \cup \{\mathbf{t} \in \wp(\sigma) \ : \ (r, s, \mathbf{t}) \in \mathcal{T}\}) \\ &= - \cap_{r \in \mathbf{r},\ s \in \mathbf{s}} \cup \{\mathbf{t} \in \wp(\sigma) \ : \ (r, s, \mathbf{t}) \in \mathcal{T}\} \\ &= - \cap_{r \in \mathbf{r},\ s \in \mathbf{s}} \cup \{t \in \sigma \ : \ (r, s, t) \in \mathcal{T}\}. \end{aligned}$$

Thus, composition in a modular qualitative calculus is uniquely defined by the triples of inconsistent base relations.

6.7.4 Abstract glue

Here we formulate the problem of defining a symbolic analogy of the semantic glue between two modular partition schemes. Assume that we have two modular qualitative calculi, $(\mathbb{A}_1, \mathbb{M}_1)$ and $(\mathbb{A}_2, \mathbb{M}_2)$, specified by their atom structures:

$$\begin{aligned} \mathfrak{S}(\mathbb{A}_1) &= \left(\sigma_1, 1'_1, \diamond, \breve{}\right), \qquad & \mathbb{M}_1 &= (\tau_1, \sigma_1, dom, cod, \iota), \\ \mathfrak{S}(\mathbb{A}_2) &= \left(\sigma_2, 1'_2, \diamond, \breve{}\right), \qquad & \mathbb{M}_2 &= (\tau_2, \sigma_2, dom, cod, \iota). \end{aligned}$$

Definition 52 (Abstract glue)**.** An *abstract glue* between $(\mathbb{A}_1, \mathbb{M}_1)$ and $(\mathbb{A}_2, \mathbb{M}_2)$ is defined as

$$G = (\sigma_g, dom_g, cod_g, \mathcal{T}_g)\,,$$

where

1) σ_g is a set of glue relation symbols disjoint with σ_1 and σ_2 (we denote the set of converse relation symbols as $\sigma_g{}^{\smile} = \{r^{\smile} \ : \ r \in \sigma_g\}$),

2) $dom_g : \sigma_g \to \tau_1$, $cod_g : \sigma_g \to \tau_2$ are surjective functions (dom_g and cod_g are naturally induced on $\sigma_g{}^{\smile}$),

3) for every $\mathfrak{s} \in \tau_1$ and $\mathfrak{t} \in \tau_2$ there exists $r \in \sigma_g$ such that $dom(r) = \mathfrak{s}$ and $cod(r) = \mathfrak{t}$,

4) $\mathcal{T}_g$ is a set of triples (r, s, t), such that

  i) $r, t \in \sigma_g \cup \sigma_g{}^{\smile}$ and $s \in \sigma_1 \cup \sigma_2$,

  ii) $dom_g(r) = dom_g(t)$, $cod_g(r) = dom(s)$, $cod(s) = cod_g(t)$,

  iii) if s is a weak identity atom, then $r \neq t$.

The *completion* of $\mathcal{T}_g$, denoted by $\mathcal{T}_g^*$, is defined as the least set with:

1. $\mathcal{T}_g \subseteq \mathcal{T}_g^*$,
2. if $r, s, t \in \sigma_1$ and $t \notin r \diamond s$, then $(r, s, t) \in \mathcal{T}_g^*$,
3. if $r, s, t \in \sigma_2$ and $t \notin r \diamond s$, then $(r, s, t) \in \mathcal{T}_g^*$,
4. for $r, s, t \in \sigma_1 \cup \sigma_2 \cup \sigma_g \cup \sigma_g{}^{\smile}$,
   if $dom(r) \neq dom(t)$ or $cod(r) \neq dom(s)$ or $cod(s) \neq cod(t)$, then $(r, s, t) \in \mathcal{T}_g^*$,
5. if $(r, s, t) \in \mathcal{T}_g$, then $(r^{\smile}, t, s), (s, t^{\smile}, r^{\smile}) \in \mathcal{T}_g^*$.

The problem is the following: given an abstract glue G between $(\mathbb{A}_1, \mathbb{M}_1)$ and $(\mathbb{A}_2, \mathbb{M}_2)$, does there exist a modular partition scheme

$$\Gamma = (\tau_1 \cup \tau_2, \sigma_1 \cup \sigma_2 \cup \sigma_g \cup \sigma_g{}^{\smile}, dom, cod, \iota),$$

such that $\mathbb{A}_1 = \mathbb{A}_\Gamma(\tau_1)$, $\mathbb{M}_1 = \mathbb{M}_\Gamma(\tau_1)$, $\mathbb{A}_2 = \mathbb{A}_\Gamma(\tau_2)$, $\mathbb{M}_2 = \mathbb{M}_\Gamma(\tau_2)$, and composition of $\mathbb{A}_\Gamma$ is the same as the one defined by $\mathcal{T}_g^*$? If the answer to this question is negative, then is it possible to impose more constraints on $\mathcal{T}_g$ to make composition modulo abstract glue a valid operation on modular qualitative calculi?

6.8 Modular partition schemes with syntactic interpretation

Qualitative calculi may arize from relations defined axiomatically, like the Region Connection Calculus (Appendix B.2). For such cases we define many-sorted constraint languages within an axiomatic theory. To combine two modular qualitative calculi, one needs to find the inconsistent triples which involve the glue relations. If relations in two many-sorted constraint languages are defined axiomatically, then the calculi generated by these constraint languages can be combined only by defining the glue predicates. Then, finding the inconsistent triples which involve the glue predicates amounts to theorem proving.

Let Θ be some axiomatic first-order theory, in which the unary predicates τ and the binary predicates σ are axiomatized. Without loss of generality we can assume that τ and σ are contained in the signature of Θ. Then the tuple

$$\Gamma = (\tau, \sigma, dom, cod, \iota, \Theta)\,,$$

where $dom, cod : \sigma \to \tau$ and $\iota : \tau \to \sigma$ is called a *modular partition scheme with syntactic (first-order) interpretation*, if

$$\begin{array}{lcl}
dom(r) = \mathfrak{s} & \Rightarrow & \Theta \models r(x,y) \to \mathfrak{s}(x) \\
cod(r) = \mathfrak{s} & \Rightarrow & \Theta \models r(x,y) \to \mathfrak{s}(y) \\
\iota(\mathfrak{s}) = r & \Rightarrow & \Theta \models \mathfrak{s}(x) \to r(x,x) \\
\sigma(\mathfrak{s},\mathfrak{t}) = \{r_1,\dots,r_k\} & \Rightarrow & \Theta \models (\mathfrak{s}(x) \wedge \mathfrak{t}(y)) \leftrightarrow (r_1(x,y) \vee \cdots \vee r_k(x,y)) \\
r_i \neq r_j \in \sigma & \Rightarrow & \Theta \models r_i(x,y) \to \neg r_j(x,y)
\end{array}$$

A triple of Θ-predicates (r,s,t) forms an inconsistent triple if

$$\Theta \models r(x,y) \wedge s(y,z) \to \neg t(x,z).$$

In the previous section we have seen that to combine two qualitative calculi, one has to specify a set of new relation symbols (glue relations) and the set of inconsistent triples which involve the glue relations. If the base relations of these calculi are defined as dyadic formulae in some theory Θ, then, to combine two calculi, one needs to define the glue relations (as dyadic formulae in Θ), and obtain the inconsistent triples by theorem proving.

6.9 Discussion

The class QCD of qualitative calculi defined in Dylla et al. (2013) is broader than the class of modular qualitative calculi. It admits algebras generated by any set of JEPD relations and allows "weaker than weak" composition and converse. However, this framework has certain disadvantages.

First, all qualitative calculi considered in Dylla et al. (2013) (Figure 3.1) are based on integral partition schemes with a weak identity. Considering the weak identity relation of a partition scheme as a distinguished element of its algebra turns the latter into a relation-type algebra and makes explicit some additional properties of such algebras, like, for example, (MQC6).

Second, the QCD framework accepts nonintegral algebras, but does not provide tools for dealing with them. Indeed, algebras generated by abstract partition schemes may not be integral. It means that they may contains two (nonempty) relations with an empty composition. At the first glance, considering such algebras seems unnecessary, because all known spatio-temporal calculi are based on integral partition schemes. However, this is not true, becasue qualitative calculi of some binary constraint languages are nonintegral (Proposition 8). Moreover, a combination modulo glue of two integral partition schemes over disjoint universes yields a nonintegral partition scheme, and thus a nonintegral calculus.

We introduced a framework for dealing with relations over a heterogeneous universe and with nonintegral algebras that such relations generate, based on modular partition schemes. They allow for combining qualitative calculi defined for different kinds of entities into a single calculus. In a modular partition scheme, each homogeneous universe can be abstracted to a weak identity atom, with an associated sort symbol. Domains and codomains of binary relations are then abstracted to functions defined on relation symbols and ranged over the sort symbols. Thus, nonintegral algebras generated by modular partition schemes allow for discriminating explicitly between kinds of entities on the symbolic level, which is not the case with integral algebras.

6.10 Conclusions

We have defined the "combination modulo glue" operation on modular partition schemes. The syntactic counterpart of the semantic glue between two universes is given by a set of glue relation symbols with their abstract domain and codomain, and a set of inconsistent triples containing the glue relations. We introduced the notion of an abstract glue and raised the question whether each abstract glue defines a valid combination operation on two modular qualitative calculi, that is, whether the combination of two modular qualitative calculi modulo an abstract glue is generated by a modular partition scheme.

We defined modular partition schemes in axiomatic first-order theories. To combine qualitative calculi based on such partition schemes, one has to define the glue predicates and find the inconsistent triples by theorem proving.

Finally, we argued that modular qualitative calculi have an advantage over QCD, because they are relation-type algebras and allow for dealing with nonintegral algebras. Modular qualitative calculi admit all qualitative spati-temporal calculi considered in Dylla et al. (2013). They have algebraic properties that allow for defining a modular structure on them.

The heterogeneous cardinal direction relations introduced in Kurata and Shi

(2009) can be modeled as a modular partition scheme. Kurata and Shi (2009) consider five kinds of spatial entities: points, general lines, horizontal lines, vertical lines and regions: $\tau = \{\mathfrak{p}, \mathfrak{gl}, \mathfrak{hl}, \mathfrak{vl}, \mathfrak{r}\}$. In modeling this as a modular partition scheme, one can use point relations between $(\mathfrak{vl}, \mathfrak{vl})$ and $(\mathfrak{hl}, \mathfrak{hl})$, square products of point relations between $(\mathfrak{p}, \mathfrak{p})$ and cardinal direction relations between $(\mathfrak{r}, \mathfrak{r})$. The glue relations between different sorts can be different. For example, between $(\mathfrak{vl}, \mathfrak{r})$ one can use glue relations "left from", "touches from the left", "intersects", "touches from the right", "right from", which are indeed JEPD on $(\mathfrak{vl}, \mathfrak{r})$. The relativization of the modular qualitative calculus of Kurata and Shi to $\mathfrak{vl}$ or $\mathfrak{hl}$ is the point calculus, to $\mathfrak{p}$ – the cardinal direction calculus (CDC) (Ligozat, 1998), to $\mathfrak{r}$ – the cardinal direction relations calculus (CDR) (Skiadopoulos and Koubarakis, 2004).

Chapter 7

A modular qualitative calculus of ontology alignments

Abstract. Qualitative calculi were shown useful in managing ontology alignments. The previously considered algebra $\mathbb{A}5$ contains taxonomical relations between classes. However, compositional inference using this algebra is sound only if we assume that classes which occur in alignments have nonempty interpretations. Moreover, $\mathbb{A}5$ covers relations only between classes. Here we introduce a novel qualitative calculus $\mathbb{A}16$, which, first, solves the limitation of the previous one, and second, incorporates all qualitative taxonomical relations that occur between individuals and concepts, including the relations "is a" and "is not". We prove that algebraic reasoning with $\mathbb{A}16$ is coherent with respect to the simple semantics of alignments.

In Chapter 4, we considered the qualitative calculus $\mathbb{A}5$ (RCC5) and discussed its limitations with respect to the simple semantics of alignments. This chapter applies the results of Chapters 5 and 6 and introduces a new qualitative calculus $\mathbb{A}16$, which solves the limitations of $\mathbb{A}5$. $\mathbb{A}16$ incorporates the relations "same as" (owl:sameAs), "different from" (owl:differentFrom), "is a" (rdf:type), "is not", "equivalent to" (owl:equivalentClass), "subsumed by" (rdfs:subClassOf), "disjoint with" (owl:disjointWith) and "partially overlaps with" in compliance with OWL semantics (Cuenca Grau et al., 2012). An earlier version of this chapter is published in (Inants and Euzenat, 2015).

The calculus $\mathbb{A}16$ is defined in Section 7.1. In Section 7.2, it is shown that $\mathbb{A}16$ allows for discriminating between unsatisfiability and incoherence of alignments, which was not possible with $\mathbb{A}5$.

7.1 The qualitative calculus of taxonomical relations

In this section, we introduce the qualitative calculus $\mathbb{A}16$, which covers all taxonomical relations. We call an ontology alignment relation *taxonomical*, if it is associated with some set-theoretic relation (predicate) R. For instance, subsumption $\sqsubseteq$ is associated with the set-theoretic inclusion $\subseteq$. A taxonomical relation holds between two ontological entities iff the relation R holds between the interpretations of these entities. We call a set-theoretic relation R *qualitative*, if, for any pair of sets (x, y), xRy is characterized by 3 parameters: whether each of the sets $x \cap y$, $x \backslash y$, $y \backslash x$ is empty or not. The relations "equivalent to", "subsumed by", "disjoint with", "same as", "different from", "partially overlaps with", "is a" and "is not" are taxonomical and qualitative (if interpreted with the simple semantics of alignments).

The simple semantics of alignments assumes a common domain of interpretation for all ontologies in a network (Section 4.2). Given an arbitrary infinite domain D, the relations "same as" and "different from" correspond to set-theoretic relations $=$ and $\neq$ on D, "equivalent to", "subsumed by", "disjoint with" and "partially overlaps with" correspond to set-theoretic relations $=$, $\subseteq$, $\|$, $\between$ on $\wp(D)$, and finally "is a" and "is not" correspond to $\in$ and $\notin$ between D and $\wp(D)$. All these relations are defined on the *universe* $D \cup \wp(D)$, denoted as $\mathcal{U}_D$. We will refer to the elements of D as *individuals*, and to the elements of $\wp(D)$ as *sets*.

We start with specifying the initial *constraint language* (relational structure) Γ. The *relational signature* of Γ is

$$\sigma = \{\equiv, \sqsubseteq, \|, \between, \in, \notin, =, \neq\}.$$

The *universe* of Γ is

$$\mathcal{U}_D = D \cup \wp(D).$$

Relation symbols in σ are interpreted over the universe. For example,

$$\equiv^\Gamma = \{(X, Y) \ : \ X, Y \in \wp(D) \text{ and } X = Y\}.$$

The constraint language Γ is not JEPD. For example, the empty set is both equivalent to itself and disjoint with itself. Thus, Γ does not generate a qualitative calculus.

Our next objective is to obtain a qualitative calculus for Γ. We will follow the methodology of Chapters 5 and 6 and construct a constraint language Γ' such that Γ is *coarser* than Γ' and relations of Γ' form a *modular partition scheme*. We want the sought-after qualitative calculus to have *strong converse*, thus we will require Γ' to be a *weakly-associative partition scheme*.

To construct Γ', we should start with finding a weakly-associative partition scheme $\mathcal{P}$ on the universe $\mathcal{U}_D$ such that Γ is coarser than $\mathcal{P}$. Using Proposition 13, we obtain a weakly-associative partition scheme $\mathcal{P}^{16}$ with 16 base relation (Table 7.1). $\mathcal{P}^{16}$ is a *heterogeneous* partition scheme, since the identity over $\mathcal{U}_D$ is a disjunctive $\mathcal{P}^{16}$-relation. The sub-identity base relations of $\mathcal{P}^{16}$ are

$$Id_1 = (\sqsubseteq \wedge \sqsupseteq \wedge \neg \parallel)^{\Gamma}, \quad Id_2 = (\sqsubseteq \wedge \sqsupseteq \wedge \parallel)^{\Gamma}, \quad \text{and} \quad Id_3 = (=^{\Gamma}).$$

The fields of these relations are

$$Fd(Id_1) \;=\; \wp(D)\backslash\{\varnothing\}, \qquad Fd(Id_2) \;=\; \{\varnothing\}, \qquad Fd(Id_3) \;=\; D.$$

These sets partition the universe $\mathcal{U}_D$ into three *kinds* of entities: *nonempty sets* noted as n, the *empty set* noted as e, and *individuals* noted as i. Thus, the set of *sorts* is $\mathcal{S} = \{n, e, i\}$.

The sought-after constraint language Γ' is obtained form $\mathcal{P}^{16}$ by chosing a relational signature, i.e., a relation symbol for each base $\mathcal{P}^{16}$-relation. The constraint language Γ' is specified in Table 7.1. The signature of Γ' is

$$\sigma' = \{=_n, \sqsubset_n, \sqsupset_n, \between, \parallel_n, \nabla_{en}, \nabla_{ne}, =_e, \in, \ni, \notin_{in}, \nabla_{ie}, \not\ni_{ni}, \nabla_{ei}, =_i, \neq_i\}.$$

The modular structure of Γ' is visualized in Figure 7.1 as a directed labeled graph.

Relation	Definition
$=_n (x, y)$	x, y are nonempty sets and $x = y$
$\sqsubset_n (x, y)$	x, y are nonempty sets and $x \subset y$
$\sqsupset_n (x, y)$	x, y are nonempty sets and $x \supset y$
$\between (x, y)$	x, y are sets and $x\backslash y,\ x \cap y,\ y\backslash x \neq \varnothing$
$\parallel_n (x, y)$	x, y are nonempty sets and $x \cap y = \varnothing$
$\nabla_{en}(x, y)$	$x = \varnothing,\ y$ is a nonempty set
$\nabla_{ne}(x, y)$	x is a nonempty set and $y = \varnothing$
$=_e (x, y)$	$x = y = \varnothing$
$\in (x, y)$	x is an individual, y is a set and $x \in y$
$\ni (x, y)$	x is a set, y is an individual and $x \ni y$
$\notin_{in} (x, y)$	x is an individual, y is a nonempty set and $x \notin y$
$\nabla_{ie}(x, y)$	x is an individual, $y = \varnothing$
$\not\ni_{ni} (x, y)$	x is a nonempty set, y is an individual and $x \not\ni y$
$\nabla_{ei}(x, y)$	y is an individual, $x = \varnothing$
$=_i (x, y)$	x, y are individuals and $x = y$
$\neq_i (x, y)$	x, y are individuals and $x \neq y$

Table 7.1: Constraint language Γ'.

Γ' generates a qualitative calculus $\mathbb{A}16$, in which the signature of Γ' is the set of atoms of $\mathbb{A}16$. Since $\mathbb{A}16$ is a modular qualitative calculus, for any $r, r' \in At(\mathbb{A}16)$, we have $cod(r) \neq dom(r') \Rightarrow r \diamond r' = \varnothing$ (Proposition 18). Thus,

n	$=_n$	$\sqsupset_n$	$\sqsubset_n$	$\between$	$\parallel_n$	$\ni$	$\not\ni_{ni}$	∇_{ne}
$=_n$	$=_n$	$\sqsupset_n$	$\sqsubset_n$	$\between$	$\parallel_n$	$\ni$	$\not\ni_{ni}$	∇_{ne}
$\sqsubset_n$	$\sqsubset_n$	$=_n\sqsupset_n\sqsubset_n\between\parallel_n$	$\sqsubset_n$	$\sqsubset_n\between\parallel_n$	$\parallel_n$	$\ni\not\ni_{ni}$	$\not\ni_{ni}$	∇_{ne}
$\sqsupset_n$	$\sqsupset_n$	$\sqsupset_n$	$=_n\sqsupset_n\sqsubset_n\between$	$\sqsupset_n\between$	$\sqsupset_n\between\parallel_n$	$\ni$	$\ni\not\ni_{ni}$	∇_{ne}
$\between$	$\between$	$\sqsupset_n\between\parallel_n$	$\sqsubset_n\between$	$=_n\sqsupset_n\sqsubset_n\between\parallel_n$	$\sqsupset_n\between\parallel_n$	$\ni\not\ni_{ni}$	$\ni\not\ni_{ni}$	∇_{ne}
$\parallel_n$	$\parallel_n$	$\parallel_n$	$\sqsubset_n\between\parallel_n$	$\sqsubset_n\between\parallel_n$	$=_n\sqsupset_n\sqsubset_n\between\parallel_n$	$\not\ni_{ni}$	$\ni\not\ni_{ni}$	∇_{ne}
$\in$	$\in$	$\in\notin_{in}$	$\in$	$\in\notin_{in}$	$\notin_{in}$	$=_i\neq_i$	$\neq_i$	∇_{ie}
$\notin_{in}$	$\notin_{in}$	$\notin_{in}$	$\in\notin_{in}$	$\in\notin_{in}$	$\in\notin_{in}$	$\neq_i$	$=_i\neq_i$	∇_{ie}
∇_{en}	∇_{en}	∇_{en}	∇_{en}	∇_{en}	∇_{en}	∇_{ei}	∇_{ei}	$=_e$

i	$=_i$	$\neq_i$	$\in$	$\notin_{in}$	∇_{ie}
$=_i$	$=_i$	$\neq_i$	$\in$	$\notin_{in}$	∇_{ie}
$\neq_i$	$\neq_i$	$=_i\neq_i$	$\in\notin_{in}$	$\in\notin_{in}$	∇_{ie}
$\ni$	$\ni$	$\ni\not\ni_{ni}$	$=_n\sqsupset_n\sqsubset_n\between$	$\sqsupset_n\between\parallel_n$	∇_{ne}
$\not\ni_{ni}$	$\not\ni_{ni}$	$\ni\not\ni_{ni}$	$\sqsubset_n\between\parallel_n$	$=_n\sqsupset_n\sqsubset_n\between\parallel_n$	∇_{ne}
∇_{ei}	∇_{ei}	∇_{ei}	∇_{en}	∇_{en}	$=_e$

e	$=_e$	∇_{en}	∇_{ei}
$=_e$	$=_e$	∇_{en}	∇_{ei}
∇_{ne}	∇_{ne}	$=_n\sqsupset_n\sqsubset_n\between\parallel_n$	$\ni\not\ni_{ni}$
∇_{ie}	∇_{ie}	$\in\notin_{in}$	$=_i\neq_i$

Table 7.2: Composition tables of the modular qualitative calculus $\mathbb{A}16$.

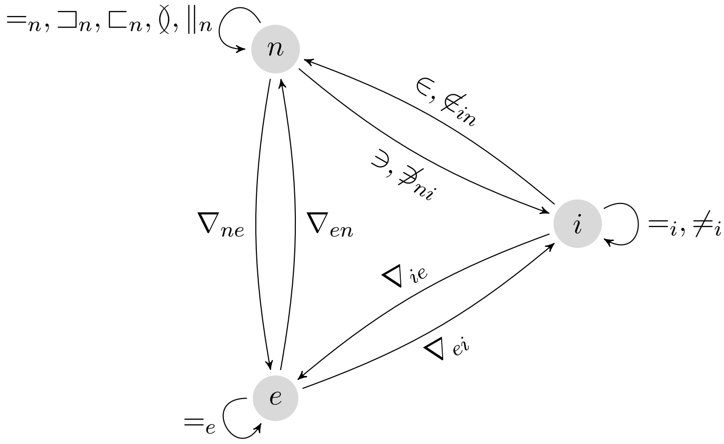


Figure 7.1: Modular structure of the constraint language Γ' with sorts e, n, i.

composition should be specified only for those atoms, for which $cod(r) = dom(r')$. Since there are three sorts, composition is specified by three composition tables, as shown in Table 7.2.

Proposition 24. $\mathbb{A}16$ *is a relation algebra.*

Proof. $\mathbb{A}16$ is (at least) a weakly-associative algebra, according to Proposition 12. The composition operation of $\mathbb{A}16$ happens to be associative – this can be checked manually. Associativity of composition makes $\mathbb{A}16$ a relation algebra. □

σ-predicate	Equivalent σ'-predicate
$\equiv$	$=_n \vee =_e$
$\sqsubseteq$	$=_n \vee =_e \vee \sqsubset_n \vee \nabla_{en}$
$\|$	$\|_n \vee =_e \vee \nabla_{en} \vee \nabla_{ne}$
$\between$	$\between$
$\in$	$\in$
$\notin$	$\notin_{in} \vee \nabla_{ie}$
$=$	$=_i$
$\neq$	$\neq_i$

Table 7.3: The σ-language Γ is coarser than the σ'-language Γ'.

Since Γ is coarser than Γ', therefore $\Gamma'_\vee$ is more expressive than $\Gamma_\vee$. The conversion from Γ to $\Gamma'_\vee$ is given in Table 7.3

Proposition 25. *Algebraic reasoning with* $\mathbb{A}16$ *is sound with respect to the simple semantics of alignments.*

Proof. Given an arbitrary simple interpretation of a network of ontologies, we can always assume that its domain D is infinite. This was the only that we made in defining the univese of interpretation $\mathcal{U}_D$ of the calculus $\mathbb{A}16$[1]. Since algebraic reasoning with $\mathbb{A}16$ is valid in each fixed model of a network of ontologies, we conclude that it is sound w.r.t. the simple semantics of alignments. □

Some ontology alignment relations from $\Gamma'_\vee$ have OWL counterparts. Since OWL semantics of such relations corresponds to the simple semantics of ontology alignment relations, we conclude that Algebraic reasoning with $\mathbb{A}16$ complies with OWL semantics. This implies that local axioms from ontologies, which express relations that exist in $\mathbb{A}16$, can be safely converted into correspondences for stronger reasoning results.

7.2 Algebraic reasoning with $\mathbb{A}16$

In Section 4.4 it was shown that the algebra $\mathbb{A}5$ does not allow for discriminating between unsatisfiability and incoherence of alignments. Proposition 26 shows that this is possible now with $\mathbb{A}16$.

Proposition 26. *Let $\mathcal{N} = (\Omega, \Lambda)$ be a network of ontologies with ontology alignment relations from $\mathbb{A}16$. If the algebraic closure of Λ has a correspondence $(\mathsf{E}_1,\ \mathsf{E}_2,\ r)$ and*

- *$dom(r) = \{e\}$, then E_1 is an incoherent class,*
- *$cod(r) = \{e\}$, then E_2 is an incoherent class,*
- *$r = \varnothing$, then $\mathcal{N}$ is unsatisfiable.*

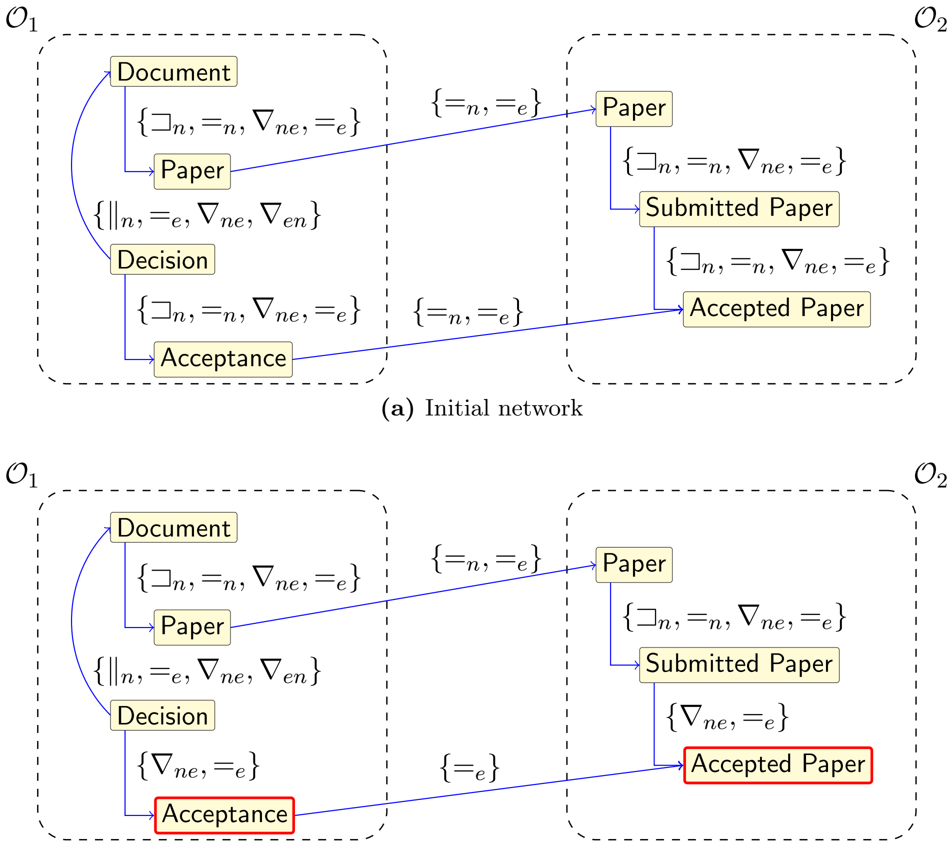
Proof. Straightforward from the definition of $\mathbb{A}16$. □

Figure 7.2 shows an example of a network of ontologies encoded in $\mathbb{A}16$ and its algebraic closure, i.e., closure under algebraic inferencing. Constraint propagation with $\mathbb{A}16$ allows for detecting that the alignment between $\mathcal{O}_1$ and $\mathcal{O}_2$ makes the classes Acceptance and Accepted Paper incoherent.

7.3 Conclusions

We applied the theoretical results about modular qualitative calculi to solve the limitations of the qualitative calculus $\mathbb{A}5$ of ontology alignment relations. This resulted in a modular qualitative calculus $\mathbb{A}16$, which covers all qualitative relations between ontology entities from the taxonomy perspective. It improves on $\mathbb{A}5$ in two ways. First, $\mathbb{A}16$ combines class-level and instance-level relations within a single calculus. Second, $\mathbb{A}16$ allows for discriminating between unsatisfiability and incoherence of a network of ontologies.

[1] This assumption prohibits "small" universes which can produce "wrong" weak operations of taxonomic relations. For example, if $|D| = 3$, then $\between \diamond \between = \{\equiv, \between\}$.



(a) Initial network

(b) Algebraic closure

Figure 7.2: An example of detecting incoherent classes in a network of ontologies, using the algebra $\mathbb{A}16$ with the modular structure $\mathcal{M}$.

The qualitative calculus $\mathbb{A}5$ is the relativization of $\mathbb{A}16$ to nonempty sets. The constraint language that generates $\mathbb{A}5$ is defined on nonempty sets. An expansion of its universe by adding one element – the empty set – breakes the integrality of its algebra. Such an expansion results in a modular qualitative calculus $\mathbb{A}8$, which is a relativization of $\mathbb{A}16$ to sets.

Chapter 8

A quasi-qualitative calculus of taxonomical relations

Abstract. Ontology alignments are often equipped with numerical attributes, which express the confidence of each correspondence. Here we define the relaxed semantics of confidence values for subsumption and equivalence relations and introduce a quasi-qualitative calculus $\mathbb{A}_{\mathsf{INTREC}}$ with numerically parametrized taxonomical relations between classes, which can be used for expressing and reasoning with weighted relations in compliance with their relaxed semantics. The calculus $\mathbb{A}_{\mathsf{INTREC}}$ contains infinitely many relations, is not closed under complementation and has weak union.

In the previous chapter, we introduced the modular qualitative calculus $\mathbb{A}16$ of taxonomical relations between individuals and classes. The relativization of $\mathbb{A}16$ to nonempty sets is the calculus $\mathbb{A}5$. In this chapter, we introduce a constraint language $\Delta_\vee$ of numerically parametrized taxonomical relations between classes, which is finer than $\mathbb{A}5$. We specify the sublanguage INTREC of $\Delta_\vee$ and define its algebra.

Qualitative representation and reasoning usually deals with non-numerical relations. However, some calculi operate with numerically parametrized relations. Such relations, as well as their calculi, are sometimes called *quasi-qualitative*. Some examples of quasi-qualitative calculi are the $2n$-star calculi (Mitra, 2002, Renz and Mitra, 2004) or the $\mathcal{OPRA}_n$ calculi of orientation relations (Moratz, 2006, Mossakowski and Moratz, 2012). The calculus $\mathbb{A}_{\mathsf{INTREC}}$, which we introduce in this chapter, is also quasi-qualitative, but, unlike the known spatio-temporal calculi, it contains infinitely many relations.

In ontology alignments, the most common relations between classes are equivalence ($\equiv$), subsumption ($\sqsubseteq$) and disjointness ($\|$). Ususally, these relations result from the ontology matching process, which is based on heuristics. Many ontology matching algorithms produce correspondences with weighted relations, that is, with an additional numerical component which expresses the confidence of the correspondence (Euzenat and Shvaiko, 2013). For example the correspondence (Car, Automobile, $\equiv_{0.8}$) expresses 80% of confidence that Automobile is equivalent to Car. An attempt to formalize the semantics of weighted ontology alignments is taken in (Atencia et al., 2012).

Ontology alignment relations between concepts may be induced based on the instance-level data. Since semantic web is an open environment with potentially invalid data, many instance-based matchers induce a relation between two concepts, if it holds for *most* instances of these concepts. The level of fault-tolerance is usually set by a threshold. This threshold may be discarded in the resulting correspondence, or may be expressed as a confidence value. To formalize this confidence measure, we introduce the *relaxed* semantics of subsumption and equivalence. For example, the correspondence $\big($Novelist, Writer, $\sqsubseteq^{0.99}\big)$ with relaxed subsumption $\sqsubseteq^{0.99}$ is interpreted as "at least 99% of novelists are writers".

The quasi-qualitative calculus $\mathbb{A}_{\mathsf{INTREC}}$ allows for expressing relaxed subsumption and equivalence. From the theoretical point of view, it is peculiar in that it contains infinitely many relations, is not closed under complementation and has weak union.

8.1 The constraint language of quasi-qualitative taxonomical relations

Let D be some countably infinite set. We will consider the set of finite nonemply subsets of D as the *universe* and denote it as $\mathcal{U}^{(D)}$, or simply $\mathcal{U}$:

$$\mathcal{U}^{(D)} = \{X \ : \ X \subseteq D \text{ and } 0 < |X| < \omega\}.$$

The set of all rational numbers not smaller than 0 and not greater than 1 will be denoted as $[0,1]_{\mathbb{Q}}$. We define a binary relational signature σ_0 as a set of ordered pairs (α, β), where $\alpha, \beta \in [0,1]_{\mathbb{Q}}$.

Further, we define a σ_0-structure Δ on the universe $\mathcal{U}$ as follows:

$$(\alpha,\beta)^{\Delta} = \left\{(X,Y) \in \mathcal{U} \times \mathcal{U} \ : \ \frac{|X \cap Y|}{|X|} = \alpha \text{ and } \frac{|X \cap Y|}{|Y|} = \beta\right\}.$$

Clearly, if $\alpha = 0$ and $\beta \neq 0$, or $\alpha \neq 0$ and $\beta = 0$, then $(\alpha,\beta)^{\Delta} = \varnothing$. This means that the relation symbols $(0,\beta)$ or $(\alpha,0)$, in which $\alpha,\beta \neq 0$, are synonyms and all denote the empty relation. We will exclude these symbols from our consideration and instead will use the symbol $\perp$ for the empty relation. For the rest of σ_0-symbols we will say (α,β) is equal to (α',β') iff $\alpha = \alpha'$ and $\beta = \beta'$.

Proposition 27. *For any $\alpha, \beta \in [0,1]_{\mathbb{Q}}$ (such that α and β are either both zero or both nonzero), the relation $(\alpha,\beta)^{\Delta}$ is not empty.*

Proof. Let $\alpha = \frac{m_1}{m_2}$ and $\beta = \frac{n_1}{n_2}$, where $m_1, m_2, n_1, n_2 \in \mathbb{N}$, $m_1 \leq m_2$ and $n_1 \leq n_2$.

Assume $m_1 < m_2$ and $n_1 < n_2$. One can always chose three pairwise disjoint sets $A, B, C \in \mathcal{U}$ such that

$$|A| = n_1(m_2 - m_1) \qquad |B| = m_1 n_1 \quad \text{and} \qquad |C| = m_1(n_2 - n_1).$$

Then we set $X = A \cup B$ and $Y = B \cup C$. Both X and Y belong to $\mathcal{U}$, and $(X, Y) \in (\alpha, \beta)^{\Delta}$.

Assume now that $m_1 = m_2$, i.e., $\alpha = 1$. Then we can choose $Y \in \mathcal{U}$ such that $|Y| = n_2$ and a subset X of Y such that $|X| = n_1$, and therefore $(X, Y) \in (\alpha, \beta)^{\Delta}$. Likewise if $n_1 = n_2$. □

Proposition 28. *Δ is an infinite strong partition scheme.*

Proof. From the definition of Δ it follows that

$$(\alpha, \beta) \neq (\alpha', \beta') \Rightarrow (\alpha, \beta)^{\Delta} \cap (\alpha', \beta')^{\Delta} = \varnothing,$$

which means that Δ is pairwise disjoint. Let us prove that it is also jointly exhaustive. Assume X, Y are two arbitrary sets from $\mathcal{U}$. Then, setting α and β as

$$\alpha = \frac{|X \cap Y|}{|X|} \quad \text{and} \quad \beta = \frac{|X \cap Y|}{|Y|},$$

we obtain that $(X, Y) \in (\alpha, \beta)^{\Delta}$.

It remains to prove that Δ is closed under converse and that the identity relation on $\mathcal{U}$ is a base relation in Δ. The closeness under converse follows from $[(\alpha, \beta)^{\Delta}]^{-1} = (\beta, \alpha)^{\Delta}$. Finally, it is easy to see that $(1, 1)$ denotes the identity relation: $(1, 1)^{\Delta} = Id_{\mathcal{U}}$. □

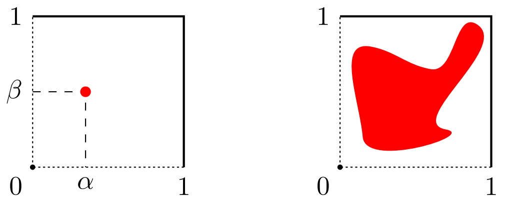
The *disjunctive expansion* of Δ (Section 3.2), denoted as $\Delta_{\vee}$, contains all unions of Γ-relations. The signature $\widehat{\sigma_0}$ of $\Delta_{\vee}$ (Definition 23) consists of all nonempty subsets of σ_0:

$$\widehat{\sigma_0} = \wp(\sigma_0).$$

An element of the signature σ_0 can be visually represented as a point on the unit square of α, β parameters (Figure 8.1a), which we will call the (α, β)-*space*. The elements of $\widehat{\sigma_0}$ correspond then to regions of the (α, β)-space, as shown in Figure 8.1b.

8.2 Quasi-qualitative relations refine the qualitative relations

Recall that in the calculus $\mathbb{A}5$, the universe is the set $\wp(D) \backslash \{\varnothing\}$, where D is a countably infinite set. In the constraint language Δ, the universe $\mathcal{U}$ consists of only *finite* nonempty subsets of D.



(a) base relation **(b)** disjunctive relation

Figure 8.1: Visual representation of quasi-qualitative relations on the (α, β)-space.

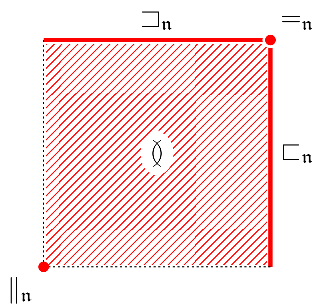
Consider the language $\Gamma = (\mathcal{U}; \equiv_\mathfrak{n}, \sqsupset_\mathfrak{n}, \sqsubset_\mathfrak{n}, \between, \|_\mathfrak{n})$ of qualitative taxonomical relations over $\mathcal{U}$. The relations $\equiv_\mathfrak{n}, \sqsupset_\mathfrak{n}, \sqsubset_\mathfrak{n}, \between, \|_\mathfrak{n}$ are defined in Table 7.1. Γ is a partition scheme on $\mathcal{U}$. Proposition 29 shows that Γ is also a partition of Δ.

Proposition 29. Δ *is a refinement of* Γ*.*

Proof. Let us show that Δ is coarser than Γ, i.e., that each (qualitative) base relation in Γ is a union of (quasi-qualitative) base relations in Δ. The map from the signature of Γ to that of $\Delta_\vee$ is given in Figure 8.2a. It is easy to check that the interpretations of the corresponding relation symbols are the same.

Γ-relation	$\Delta_\vee$-relation
$\equiv_\mathfrak{n}$	$(1, 1)$
$\sqsupset_\mathfrak{n}$	$\{(\alpha, 1) \ : \ 0 < \alpha < 1\}$
$\sqsubset_\mathfrak{n}$	$\{(1, \beta) \ : \ 0 < \beta < 1\}$
$\between$	$\{(\alpha, \beta) \ : \ 0 < \alpha, \beta < 1\}$
$\|_\mathfrak{n}$	$(0, 0)$

(a) signature mapping



(b) visualization on the (α, β)-space

Figure 8.2: The constraint language Γ of qualitative relations is a sublanguage of the constraint language of quasi-qualitative relations $\Delta_\vee$.

Since both Γ and Δ are partitions on the same universe and Γ is coarser than Δ, we conclude that Δ is a refinement of Γ. □

The base qualitative relations are visualized on the (α, β)-space in Figure 8.2b.

8.3 The sublanguage INTREC

The signature $\widehat{\sigma_0}$ of $\Delta_\vee$ contains infinite sets, thus is not useful for representing the infinite disjunctions of Δ-relations. In this section, we will solve this problem by restricting ourselves to certain kinds of $\Delta_\vee$-relations and by introducing new relation symbols for them.

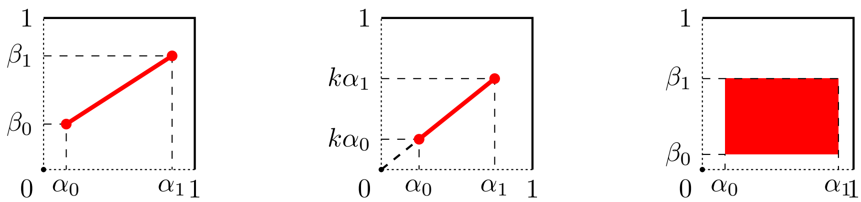
Let us introduce abbreviations for some relation symbols in $\widehat{\sigma_0}$:

$$\begin{aligned} \mathsf{INT}(\alpha_0,\beta_0,\alpha_1,\beta_1) &:= \{(\alpha_0+t(\alpha_1-\alpha_0),\beta_0+t(\beta_1-\beta_0)) \;:\; t\in[0,1]_{\mathbb{Q}}\} \\ \mathsf{REC}(\alpha_0,\beta_0,\alpha_1,\beta_1) &:= \{(\alpha,\beta) \;:\; \alpha_0\le\alpha\le\alpha_1 \text{ and } \beta_0\le\beta\le\beta_1\} \end{aligned}$$

The relation symbols $\mathsf{INT}(\alpha_0,\beta_0,\alpha_1,\beta_1)$, where $\alpha_0,\beta_0,\alpha_1,\beta_1\in[0,1]_{\mathbb{Q}}$, correspond to intervals on the (α,β)-space, as shown in Figure 8.3a. We will call them *interval relations* (not to confuse with Allen's temporal intervals). A special kind of interval relations are those, for which the $\frac{\beta}{\alpha}$ ratio is constant for all constituent base relations (α,β), as shown in Figure 8.3b. On the (α,β)-space these relations lie on a line which passes through the point $(0,0)$. We call them *0-interval relations*. These relations can be parametrized by three values: α_0, α_1 and $k=\frac{\beta}{\alpha}$.

$$\mathsf{INT}_0(\alpha_0,\alpha_1,k) := \{(\alpha,k\alpha) \;:\; \alpha_0\le\alpha\le\alpha_1\}$$

The relation symbols $\mathsf{REC}(\alpha_0,\beta_0,\alpha_1,\beta_1)$, where $\alpha_0,\beta_0,\alpha_1,\beta_1\in[0,1]_{\mathbb{Q}}$, correspond to rectangles on the (α,β)-space, the edges of which are parallel to those of the unit square (Figure 8.3c). We call them *rectangle relations*.



(a) interval **(b)** 0-interval **(c)** rectangle

Figure 8.3: Visual representation of INTREC relations.

The INTREC sublanguage of $\Delta_\vee$ consists of the base $\Delta_\vee$-relations, i.e., the point relations, the 0-interval relations (INT_0) and the rectangle relations (REC):

$$\mathsf{INTREC} = \left(\sigma_1,\ \mathcal{U},\ \cdot^{\Delta_\vee}\right), \tag{8.1}$$

where

$$\begin{aligned} \sigma_1 = &\left\{(\alpha,\beta) \;:\; \alpha=0 \Leftrightarrow \beta=0\right\} \\ &\cup \left\{\mathsf{INT}_0(\alpha_0,\alpha_1,k) \;:\; \alpha_0<\alpha_1 \text{ and } k\alpha_1\le 1\right\} \\ &\cup \left\{\mathsf{REC}(\alpha_0,\beta_0,\alpha_1,\beta_1) \;:\; \alpha_0\le\alpha_1,\quad \beta_0\le\beta_1,\quad \alpha_0=\alpha_1\Rightarrow\beta_0\ne\beta_1,\right. \\ &\qquad\qquad \left.\alpha_0+\alpha_1>0,\quad \beta_0+\beta_1>0\right\}, \end{aligned}$$

with $\alpha,\beta,\alpha_0,\beta_0,\alpha_1,\beta_1\in[0,1]_{\mathbb{Q}}$ and $k\in(0,+\infty)_{\mathbb{Q}}$. Different relation symbols in σ_1 denote different relations.

8.4 The algebra of INTREC

In this section, I introduce the algebra generated by the constraint language INTREC.

Proposition 30. INTREC *is closed under arbitrary (finite or infinite) nonempty intersections, that is, if* $(R_i)_{i\in I} \in$ INTREC *and* $\cap_{i\in I} R_i \neq \varnothing$, *then* $\cap_{i\in I} R_i \in$ INTREC.

Proof. The function $\cdot^{\Delta_\vee} : \wp(\sigma) \to \Delta_\cup$ is an isomorphism between the Boolean algebras $\wp(\sigma)$ and $\Delta_\cup$, thus it preserves all (finite or infinite) intersections. Hence, the subset INTREC of $\Delta_\cup$ is closed under arbitrary intersections iff so is the corresponding (under the isomorphism) subset σ_1 of $\wp(\sigma)$. Since the set of point, 0-interval and rectangle relation symbols is closed under arbitrary nonempty intersections, so is the set of INTREC-relations. □

Proposition 31. *For any* $R, S \in$ INTREC, $R \circ S \neq \varnothing$.

Proof. It is enough to prove this for all point relations, since all INTREC-relations are unions of point relations. Assume $R = (\alpha, \beta)^\Delta$ and $S = (\alpha', \beta')^\Delta$. Recall that $(0,0)^\Delta$ is the disjointness relation on $\mathcal{U}$. If $\alpha = \beta = 0$ or $\alpha' = \beta' = 0$, then $(\alpha,\beta)^\Delta \circ (\alpha',\beta')^\Delta \neq \varnothing$. Assume $\alpha, \beta, \alpha', \beta' \neq 0$. Let $\alpha = \frac{m_1}{m_2}$, $\beta = \frac{m_3}{m_4}$, $\alpha' = \frac{m_5}{m_6}$, $\beta' = \frac{m_7}{m_8}$. We can choose three pairwise disjoint sets A, B, Y, where $A, B \in \mathcal{U} \cup \{\varnothing\}$ and $Y \in \mathcal{U}$, such that

$$|A| = m_2 \cdot m_3 \cdot m_6 \cdot m_7 - m_1 \cdot m_3 \cdot m_6 \cdot m_7,$$
$$|B| = m_1 \cdot m_4 \cdot m_5 \cdot m_8 - m_1 \cdot m_4 \cdot m_5 \cdot m_7,$$
$$|Y| = m_1 \cdot m_4 \cdot m_6 \cdot m_7.$$

Then we can chose two subsets $C, D \subseteq Y$, such that

$$|C| = m_1 \cdot m_3 \cdot m_6 \cdot m_7,$$
$$|D| = m_1 \cdot m_4 \cdot m_5 \cdot m_7.$$

By construction, C is disjoint with A and D is disjoint with B. We set $X = A \cup C$ and $Z = B \cup D$. Both X and Z are nonempty and belong to $\mathcal{U}$. By construction, the intersections of X and Z with Y are equal to C and D respectively. It is easily verified now that $(X,Y) \in (\alpha,\beta)^\Delta$ and $(Y,Z) \in (\alpha',\beta')^\Delta$. From that we obtain $(X,Z) \in (\alpha,\beta)^\Delta \circ (\alpha',\beta')^\Delta = R \circ S$, therefore $R \circ S \neq \varnothing$. □

Definition 53 (Weak composition of INTREC-relations)**.** For $R, S \in$ INTREC, their *weak composition* is defined as

$$R \diamond S = \cap\{T \in \text{INTREC} \; : \; R \circ S \subseteq T\}.$$

$R \diamond S$ is nonempty, since $\varnothing \neq R \circ S$ (Proposition 31) and $R \circ S \subseteq R \diamond S$, therefore, according to Proposition 30, $R \diamond S \in \mathsf{INTREC}$. Thus, weak composition is an operation on INTREC.

INTREC is not closed under union. For example, the union of two overlapping rectangle relations is not an INTREC-relation. We define *weak union* on INTREC as the least INTREC-relation which contains the ordinary union.

Definition 54 (Weak union of INTREC-relations)**.** The *weak union* of $R, S \in \mathsf{INTREC}$, denoted as $R \cup_w S$, is the intersection of all $T \in \mathsf{INTREC}$, for which $R \subseteq T$ and $S \subseteq T$:

$$R \cup_w S = \cap\{T \in \mathsf{INTREC} \ : \ R, S \subseteq T\}.$$

Weak union of INTREC-relations induces (through the function $\cdot^{\sigma_1}$) an operation on the set σ_1, which we also call weak union and denote by the same symbol $\cup_w$.

INTREC is closed under (strong) converse. The converse $^{-1}$ of INTREC-relations induces an operation $\breve{}$ on the set σ_1 of relation symbols.

$$(\alpha, \beta)^{\smile} = (\beta, \alpha)$$
$$\mathsf{INT}_0(\alpha_0, \alpha_1, k)^{\smile} = \mathsf{INT}_0(k\alpha_0, k\alpha_1, k^{-1})$$
$$\mathsf{REC}(\alpha_0, \beta_0, \alpha_1, \beta_1)^{\smile} = \mathsf{REC}(\beta_0, \alpha_0, \beta_1, \alpha_1)$$

Now we can define the algebra generated by INTREC as

$$\mathbb{A}_{\mathsf{INTREC}} = (\sigma_1,\ \cup_w,\ \cap,\ \diamond,\ \breve{},\ \bot,\ \mathsf{REC}(0,0,1,1),\ (1,1)), \tag{8.2}$$

where $\bot$ is the zero element, $\mathsf{REC}(0,0,1,1)$ is the unit element and $(1,1)$ is the identity element. $\mathbb{A}_{\mathsf{INTREC}}$ is not a relation-type algebra, since it does not have the complementation operation. The other peculiarity of $\mathbb{A}_{\mathsf{INTREC}}$ is that it has weak union.

8.5 Composition of base INTREC-relations

The following proposition shows that the (weak) composition of base $\Delta_\vee$-relations is an interval $\Delta_\vee$-relation.

Proposition 32. *If* $0 < \alpha, \beta, \alpha', \beta' < 1$*, then*

$$(\alpha, \beta) \diamond (\alpha', \beta') = \mathsf{INT}_0(\alpha_0'', \alpha_1'', k), \tag{8.3}$$

where

$$\alpha_0'' = \frac{\alpha}{\beta} max\left(\alpha' + \beta - 1, 0\right),$$
$$\alpha_1'' = min\left[1, \frac{\alpha\alpha'}{\beta\beta'}, \alpha\left(min(1, \frac{\alpha'}{\beta}) + min(\frac{\alpha'}{\beta}\frac{1-\beta'}{\beta'}, \frac{1-\alpha}{\alpha})\right)\right],$$
$$k = \frac{\beta\beta'}{\alpha\alpha'}.$$

Proof. Assume $A(\alpha,\beta)B(\alpha',\beta')C$. Denote $|A\cap B|$ by x. Then $|A\backslash B| = \frac{1-\alpha}{\alpha}x$. Further, denote $|A\cap B\cap C|$ by y and $|(A\cap C)\backslash B|$ by z (Figure 8.4). Then

$$|(B\cap C)\backslash A| = \frac{\alpha'}{\beta}x - y,$$
$$|C\backslash(A\cup B)| = \frac{\alpha'}{\beta}\frac{1-\beta'}{\beta'}x - z.$$

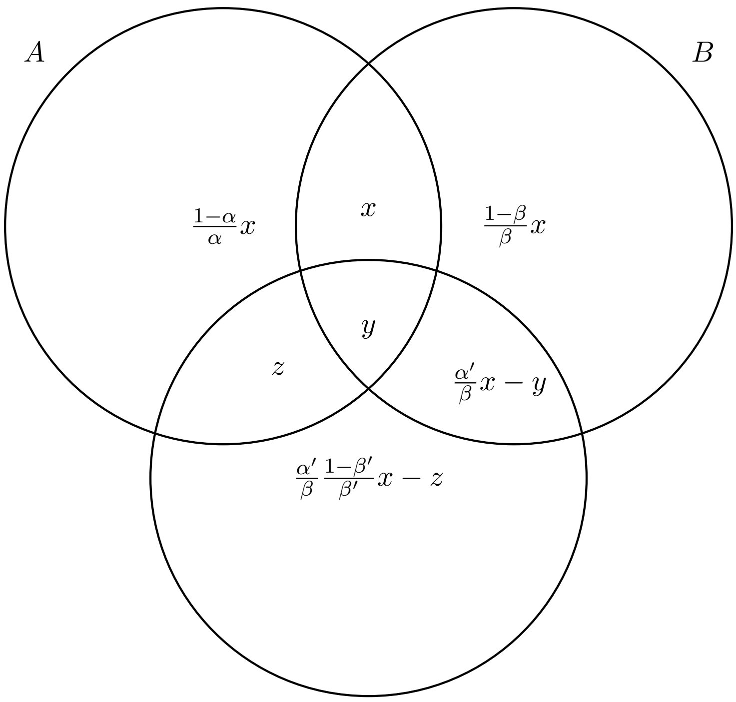


Figure 8.4: Composition of base Δ-relations.

On one hand, $0 \le z \le \frac{1-\alpha}{\alpha}x$ and $0 \le y \le x$. On the other hand,

$$0 \le \frac{\alpha'}{\beta}x - y \le \frac{1-\beta}{\beta}x \quad \text{and}$$
$$\frac{\alpha'}{\beta}\frac{1-\beta'}{\beta}x - z \ge 0.$$

Thus, we obtain

$$\begin{cases} max\left(\frac{\alpha'+\beta-1}{\beta}, 0\right)x \le y \le min\left(1, \frac{\alpha'}{\beta}\right), \\ 0 \le z \le min\left(\frac{\alpha'}{\beta}\frac{1-\beta'}{\beta'}, \frac{1-\alpha}{\alpha}\right). \end{cases} \qquad (*)$$

Let us denote

$$\alpha'' = |\frac{A\cap C}{A}|, \quad \beta'' = |\frac{A\cap C}{C}|.$$

We have that

$$\alpha'' = \frac{z+y}{x}\alpha, \quad \beta'' = \frac{z+y}{x}\frac{\beta\beta'}{\alpha'}.$$

From here we obtain that $\beta'' = \frac{\beta\beta'}{\alpha\alpha'}\alpha''$.

Denote $\frac{z+y}{x}\alpha$ by t. Let us evaluate t. z and y are free variables constrained by (*) only. Thus,

$$\frac{\alpha}{\beta} max\left(\alpha' + \beta - 1, 0\right) \le t \le \alpha \left(min(1, \frac{\alpha'}{\beta}) + min(\frac{\alpha'}{\beta}\frac{1-\beta'}{\beta'}, \frac{1-\alpha}{\alpha}) \right).$$

On the other hand, $t \le 1$ and $t\frac{\beta\beta'}{\alpha\alpha'} \le 1$, therefore

$$\begin{cases} t \ge \frac{\alpha}{\beta} max\left(\alpha' + \beta - 1, 0\right), \\ t \le min\left[1, \frac{\alpha\alpha'}{\beta\beta'}, \alpha\left(min(1, \frac{\alpha'}{\beta}) + min(\frac{\alpha'}{\beta}\frac{1-\beta'}{\beta'}, \frac{1-\alpha}{\alpha})\right)\right]. \end{cases} \qquad (**)$$

The base $\Delta_\vee$-relations that can hold between A and C are of the form $(t|kt)$, where $k = \frac{\beta\beta'}{\alpha\alpha'}$ and t satisfies (**). Moreover, since (**) is the strongest constraint on t, for any t satisfying (**) (t, kt) belongs to the composition of (α, β) and (α', β'). Thus, $(\alpha, \beta) \circ (\alpha', \beta')$ is equal to the interval $\Delta_\vee$-relation specified in the statement of the proposition.

□

Proposition 33. *If* $0 < \alpha, \beta < 1$, *then*

$$(0,0) \diamond (\alpha, \beta) = \mathsf{REC}(0, 0, 1, 1-\beta). \qquad (8.4)$$

Proof. Let $|A \cap C| = x$ and $|B \cap C| = y$ (Figure 8.5). We need to evaluate those α', β', for which $A(\alpha', \beta')C$. The only constraints on α', β' that follow from $A(0,0)B(\alpha,\beta)C$ are:

$$\begin{cases} 0 \le y \le |C|, \\ 0 \le x \le |C| - y, \\ \alpha = \frac{y}{|B|}, \\ \beta = \frac{y}{|C|}. \\ \alpha' = \frac{x}{|A|}, \\ \beta' = \frac{x}{|C|}. \end{cases}$$

From this it follows that $0 \le \alpha' \le 1$ and $0 \le \beta' \le 1 - \beta$.

□

8.6 Composition of disjunctive INTREC-relations

In this section, we prove that weak composition of INTREC-relations distributes over weak union. This result can be used to obtain formulas for weak composition of disjunctive INTREC-relations, based on the formulas for composing base relations.

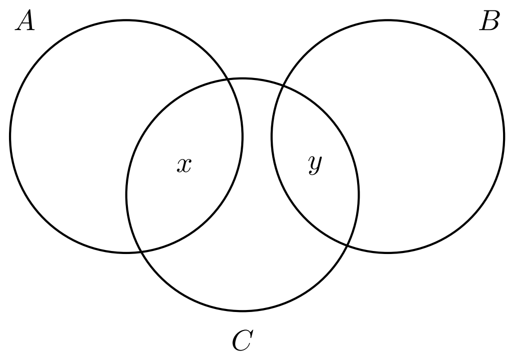


Figure 8.5: Composition of $(0, 0)$ with other Δ-relations.

Proposition 34. *For any* $R, S, T \in \mathsf{INTREC}$, *if* $R \cup S \in \mathsf{INTREC}$, *then*

$$(R \cup S) \diamond T = (R \diamond T) \cup_w (S \diamond T). \tag{8.5}$$

Proof. Due to the monotonicity of weak composition, $R \diamond T \subseteq (R \cup S) \diamond T$ and $S \diamond T \subseteq (R \cup S) \diamond T$, thus

$$(R \diamond S) \cup (S \diamond T) \subseteq (R \cup S) \diamond T. \tag{*}$$

From (*) and the definition of weak union we obtain

$$(R \diamond S) \cup_w (S \diamond T) \subseteq (R \cup S) \diamond T.$$

The opposite inclusion $(R \cup S) \diamond T \subseteq (R \diamond S) \cup_w (S \diamond T)$ follows from

$$(R \cup S) \circ T = (R \circ T) \cup (S \circ T) \subseteq (R \diamond T) \cup_w (S \diamond T)$$

and the definition of weak composition. □

8.7 Applications of INTREC in ontology alignments

Ontology alignment relations between concepts may be induced based on the instance-level data. Since semantic web is an open environment with potentially invalid data, many instance-based matchers induce a relation between two concepts, if it holds for *most* instances of these concepts. The level of fault-tolerance is usually set by a threshold. This threshold may be discarded in the resulting correspondence, or may be expressed as a confidence value. To formalize this confidence measure, we introduce the *relaxed* semantics of subsumption and equivalence.

(A, B, $\sqsubseteq^\eta$)	At least $\eta \cdot 100\%$ of instances of A are instances of B
(A, B, $\equiv^\eta$)	At least $\eta \cdot 100\%$ of instances of $A \vee B$ are instances of $A \wedge B$

Correspondences with relaxed subsumption or equivalence can be expressed, or estimated, by $\mathbb{A}_{\mathsf{INTREC}}$. The relation $\sqsubseteq^{\eta}$, where $\eta \in [0,1]_{\mathbb{Q}}$, corresponds to the INTREC-relation $\mathsf{REC}(\eta, 0, 1, 1)$. The relation $\equiv^{\eta}$ can be defined as the intersection of $\sqsubseteq^{\eta}$ and $\sqsupseteq^{\eta}$.

The universe of INTREC can be expanded to include the emptyset in exactly the same way as in the case of $\mathbb{A}_5$. This results in a modular qualitative calculus with two sorts: $\mathfrak{n}$ (nonempty sets) and $\mathfrak{e}$ (empty set), with INTREC-relations on $\mathfrak{n}$, the trivial $\nabla_{\mathfrak{ee}}$ relation on $\mathfrak{e}$, the relations $\nabla_{\mathfrak{en}}$ between $\mathfrak{e}$ and $\mathfrak{n}$ and the relation $\nabla_{\mathfrak{ne}}$ between $\mathfrak{n}$ and $\mathfrak{e}$. The relations $\nabla_{\mathfrak{ee}}$, $\nabla_{\mathfrak{en}}$ and $\nabla_{\mathfrak{ne}}$ are defined in Table 7.1. Composition of $\nabla_{\mathfrak{ne}}$ with $\nabla_{\mathfrak{en}}$ is equal to the universal relation on $\mathfrak{n}$, that is, to $\mathsf{REC}(0, 0, 1, 1)$.

Figure 8.2b shows that $\Delta_{\vee}$ is a refinement of the constraint language that generates $\mathbb{A}5$. In other words, $\mathbb{A}5$-relations "partition" the constraint language $\Delta_{\vee}$ into finitely many qualitative relations. This partitioning can be done differently, as shown in Figure 8.6, by consolidating $\Delta_{\vee}$ into relaxed qualitative relations.

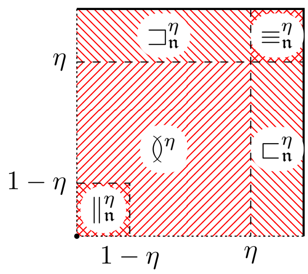


Figure 8.6: Relaxed qualitative relations.

8.8 Conclusions

The algebra $\mathbb{A}_{\mathsf{INTREC}}$, introduced in this chapter, can be used to express relaxed subsumption and equivalence, and compose alignments with such relations. For example, the relaxed subsumption $\sqsubseteq^{\eta}$ is expressed as $\{\mathsf{REC}(\eta, 0, 1, 1), \nabla_{\mathfrak{en}}, \nabla_{\mathfrak{ee}}\}$.

$\mathbb{A}_{\mathsf{INTREC}}$ is not a relation-type algebra, since it is not closed under complementation. However, it is a *constraint algebra* (Definition 26), thus can be used by any constraint propagation algorithm which supports this class of algebras.

We specified the composition operation for base INTREC-relations and showed that it can be specified for disjunctive INTREC-relations by distributing the composition on atoms and consolidating the result with weak union (Proposition 34). The complete specification of operations in $\mathbb{A}_{\mathsf{INTREC}}$ remains for future work.

Chapter 9

Conclusions and future work

This dissertation contributes to two rather independent fields: qualitative representation and reasoning and ontology matching.

Revisiting the scope of the qualitative calculi paradigm

The algebraic approach to reasoning about commonsense knowledge has been studied chiefly within the scope of reasoning about time or space. In qualitative spatio-temporal reasoning, the framework for applying this approach is called "qualitative calculi". We have shown that this framework is useful beyong the spatio-temporal domain and deserves to be considered within a broader scope of knowledge representation. We introduced two novel qualitative calculi, $\mathbb{A}16$ and $\mathbb{A}_{\mathsf{INTREC}}$, of ontology alignment relations.

Dealing with relations over heterogeneous universes

Some applications of qualitative representation and reasoning differentiate between kinds of entities. Heterogeneity of entities is crucial in ontology alignments. It is the case even in some qualitative spatial models. The existing frameworks of qualitative calculi have only one representation primitive: relations. We introduced the class of modular qualitative calculi with a subclass of weakly-associative qualitative calculi, which add one more primitive called sorts. A modular qualitative calculus consists of two symbolic component: a conventional algebra of relations and an aditional modular structure, which introduces abstract sorts and connects them with relation symbols by abstract domain and codomain functions.

Combination of qualitative calculi over different universes

Modular qualitative calculi defined over different universes can be combined into a single modular qualitative calculus, provided some "glue" relations between these universes. This operation, called "combination modulo glue", is defined on the semantic level. If the semantic component is defined syntactically (the case of syntactic interpretation), then the symbolic component can be obtained by theorem proving, specifically by revealing inconsistent triples that contain the glue relations. In the Alignment API, only the symbolic component of qualitative calculi is implemented. To combine two symbolic modular qualitative calculi, the set of inconsistent triples containing the glue relations should be provided as an input.

Comparison of modular qualitative calculi with existing frameworks

Even in the spatio-temporal context, there are several frameworks of qualitative calculi. We have shown that these frameworks can be characterized in terms of classes of partition schemes. The relation between these frameworks can be seen as the relation between corresponding classes of partition schemes (Figure 9.1). Qualitative calculi of Ligozat and Renz (with semi-strong representation) are algebras generated by non-associative partition schemes. Qualitative calculi of Westphal et. al. are algebras generated by strong partition schemes. Qualitative calculi of Dylla et. al. (with semi-strong interpretation) are algebras generated by an arbitrary set of JEPD relations – the so-called abstract partition schemes.

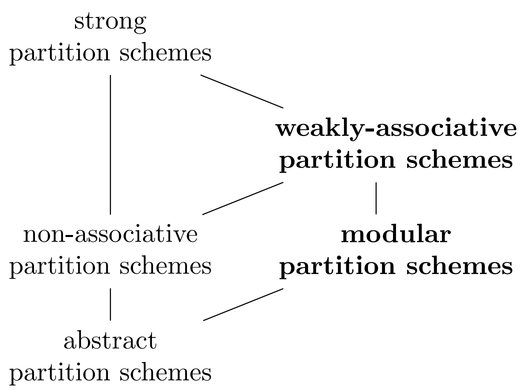


Figure 9.1: A diagram of partition scheme classes.

Quasi-qualitative calculi may be infinite

In the qualitative reasoning community, the algebraic approach is widely considered to be applicable only for a finite number of relations, be qualitative or quasi-qualitative. The quasi-qualitative calculus $\mathbb{A}_{\mathsf{INTREC}}$ defined in Chapter 8 is remarkable in that it is an instance of an infinite quasi-qualitative calculus. Moreover, unlike most known qualitative spatio-temporal calculi, its relations are not closed under union.

Benefits of qualitative reasoning in ontology alignments

We applied the theoretical results about modular qualitative calculi to solve the limitations of the qualitative calculus $\mathbb{A}5$ of ontology alignment relations. This resulted in a modular qualitative calculus $\mathbb{A}16$, which covers all qualitative relations between ontology entities from the taxonomy perspective. It improves on $\mathbb{A}5$ in two ways. First, $\mathbb{A}16$ combines class-level and instance-level relations within a single calculus. Second, $\mathbb{A}16$ allows for discriminating between unsatisfiability and incoherence of a network of ontologies.

The calculus $\mathbb{A}_{\mathsf{INTREC}}$ also contributes to ontology matching. It can be used in correspondences with relaxed subsumption and equivalence relations.

Qualitative calculus of a constraint language

Constraint languages are a more general way of specifing CSPs over infinite universes than qualitative calculi. A constraint language may not be JEPD. We have shown that every binary constraint language Γ falls into (is coarser than) some weakly-associative partition scheme Γ'. This means that the modular qualitative calculus $\mathbb{A}_{\Gamma'}$ can be used for reasoning on instances of CSP(Γ) for any binary constraint language Γ. Moreover, we have shown that it is always possible to chose Γ' in a way that $\mathbb{A}_{\Gamma'}$ is a semi-associative algebra.

9.1 Future work

The existing frameworks for qualitative calculi make it clear how to design a calculus based on a relational model for homogeneous entities. However, they provide no means for dealing with *entity models*. This is the added value of modular qualitative calculi, which make it possible to discriminate between homogeneous relations – those defined between entities of the same kind – and heterogenous, or glue relations, which are defined between entites of different kinds. Heterogeneous relations establish a connection between different entity models. From a more general perspective, this can be seen as a step in expanding the algebraic reasoning approach from commonsense temporal or spatial concepts to arbitrary ontological concepts. From this perspective, the notion of a sort, introduced in qualitative calculi, resembles the notion of a class in ontologies.

We raised the problem of establishing the abstract counterpart for the operation of combination modulo glue. The abstract glue is given by a set of so-called inconsistent triples. The question whether *any* abstract glue that meets certain symbolic constraints, corresponds to some actual semantic glue, is open and will be a subject of future work.

We grounded modular partition schemes not only in concrete binary relations, but also in binary predicates defined in axiomatic first-order theories. This opens the perspective of combining mereological part-whole relations between ontological concepts with taxonomical relations defined within the finitely axiomatized von Neumann-Bernays-Gödel (NGB) set theory.

I plan to expand the framework for qualitative calculi on predicates defined in non-classical logics, such as intuitionistic logic. The methodology for doing this is based on the notion of the Lindenbaum-Tarski algebra of a logical theory, the elements of which are classes of equivalent (congruent) sentences. For example, Lindenbaum-Tarski algebras of first-order propositional theories are complete Boolean algebras, whereas for theories in intuitionistic logic they are Heyting algebras.

The classical accounts of concrete binary relations assume that relations are Boolean, i.e., for each pair of individuals a given relation is either true or false. Among non-classical accounts of binary relations are *fuzzy relation algebras* (Kawahara and Furusawa, 1999). Unlike Boolean relation algebras, fuzzy relation algebras are not Boolean but equipped with semi-scalar multiplication. Building a framework for algebraic reasoning with fuzzy relations is another direction of future work.

Appendices

Appendix A

Implementation

Algorithm 1: Partition

Input: A set X and a set Sub of subsets of X

Output: A partiton $\mathcal{P}$ of X, such that Sub is coarser than $\mathcal{P}$

$\mathcal{P} \longleftarrow Sub \cup (X \setminus \cup Sub)$
while *iterate* **do**
    $iterate \longleftarrow false$
    for $P_1 \neq P_2 \in \mathcal{P}$ **do**
        $A \longleftarrow P_1 \cap P_2$
        if $A \neq 0$ **then**
            $iterate \longleftarrow true$
            $B \longleftarrow P_1 \setminus P_2;\ C \longleftarrow P_2 \setminus P_1$
            if $B, C = 0$ **then**
                remove P_2 form $\mathcal{P}$
            else if $B = 0$ **then**
                remove P_2 from $\mathcal{P}$ and insert C into $\mathcal{P}$
            else if $C = 0$ **then**
                remove P_1 from $\mathcal{P}$ and insert B into $\mathcal{P}$
            else
                remove P_1, P_2 from $\mathcal{P}$ and insert A, B, C into $\mathcal{P}$

Algorithm 2: Intersect partitions of the same set (IntersectPart)

Input: Partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of the same set
Output: A partition $\mathcal{P}_3$

$\mathcal{P}_3 \longleftarrow \varnothing$
for $A \in \mathcal{P}_1,\ B \in \mathcal{P}_2$ **do**
  $C \longleftarrow A \cap B$
  if $C \neq \varnothing$ **then**
    insert C into $\mathcal{P}_3$

Algorithm 3: Weakly-associative partition scheme (SortPartScheme)

Input: A set U and a set Rel of binary relation on U
Output: A weakly-associative partiton scheme $\mathcal{P}$ coarser than Rel

$Rel \longleftarrow Rel \cup Rel^{-1} \cup Id_U$
$\mathcal{P} \longleftarrow Partition(U \times U, Rel)$
$\mathcal{I} \longleftarrow \{I \in \mathcal{P} \ : \ I \cap Id_U \neq \varnothing\}$ // identity atoms of $\mathcal{P}$
$\mathcal{P} \longleftarrow IntersectPart(\mathcal{P}, \{Fd(I) \times Fd(J) \ : \ I, J \in \mathcal{I}\})$

Algorithm 4: Combine modular partition schemes (CombinePart)

Input: Partition schemes $\mathcal{P}_1$ and $\mathcal{P}_2$ over U_1 and U_2, and a set $Glue$ of binary relations between U_1 and U_2
Output: A combined partition scheme $\mathcal{P}_3$

$GluePart \longleftarrow Partition(U_1 \times U_2, Glue)$
$\mathcal{P}_3 \longleftarrow \mathcal{P}_1 \cup \mathcal{P}_2 \cup GluePart \cup GluePart^{-1}$
$\mathcal{I} \longleftarrow \{I \in \mathcal{P}_1 \ : \ I \cap Id_{U_1} \neq \varnothing\} \cup \{I \in \mathcal{P}_2 \ : \ I \cap Id_{U_2} \neq \varnothing\}$
$\mathcal{U} \longleftarrow \{Fd(I) \ : \ I \in \mathcal{I}\}$
$\mathcal{P}_3 \longleftarrow IntersectPart(\mathcal{P}_3, \{U_i \times U_j \ : \ U_i, U_j \in \mathcal{U}\})$

Algorithm 5: Generate a semi-associative partition scheme (SemiAssocPart)

Input: A set U and a set Rel of binary relation on U
Output: A weakly-associative partiton scheme $\mathcal{P}$ with semi-associative composition

$\mathcal{P} \longleftarrow SortPartScheme(U, Rel)$
$D \longleftarrow \{Dom(R) \ : \ R \in \mathcal{P}\}$
$\mathcal{U} \longleftarrow Partition(U, D)$
$\mathcal{P} \longleftarrow IntersectPart(\mathcal{P}, \{U_i \times U_j \ : \ U_i, U_j \in \mathcal{U}\})$

Appendix B

Some spatio-temporal qualitative calculi

B.1 Interval calculus

The universe of Allen's temporal interval calculus is the set

$$U = \{(x_1, x_2)\ :\ x_1, x_2 \in X \text{ and } x_1 < x_2\},$$

where $X = \mathbb{R}$ or $X = \mathbb{Q}$. The base relations of the interval calculus are defined in Table B.1.

Relation	Interpretation	Definition
$x\ \mathsf{p}\ y$	*x precedes y*	$x_1 < x_2 < y_1 < y_2$
$y\ \mathsf{pi}\ x$	*y is preceded by x*	
$x\ \mathsf{m}\ y$	*x meets y*	$x_1 < x_2 = y_1 < y_2$
$y\ \mathsf{mi}\ x$	*y is met by x*	
$x\ \mathsf{o}\ y$	*x overlaps y*	$x_1 < y_1 < x_2 < y_2$
$y\ \mathsf{oi}\ x$	*y is overlapped by x*	
$x\ \mathsf{s}\ y$	*x starts y*	$x_1 = y_1 < x_2 < y_2$
$y\ \mathsf{si}\ x$	*y is started by x*	
$x\ \mathsf{d}\ y$	*x during y*	$y_1 < x_1 < x_2 < y_2$
$y\ \mathsf{di}\ x$	*y contains x*	
$x\ \mathsf{f}\ y$	*x finishes y*	$y_1 < x_1 < x_2 = y_2$
$y\ \mathsf{fi}\ x$	*y is finished by x*	
$x\ \mathsf{eq}\ y$	*x equals y*	$x_1 = y_1$ and $x_2 = y_2$

Table B.1: Definition of Allen's basic relations between an interval $x = (x_1, x_2)$ and an interval $y = (y_1, y_2)$.

B.2 Region connection calculus

Region Connection Calculus (RCC) is based on a primitive "connectedness" predicate $\mathsf{C}(x, y)$. The relation $\mathsf{C}(x, y)$ is required to be reflexive and symmetric.

(*RCC1*) $\forall x\ \mathsf{C}(x, x)$

(*RCC2*) $\forall x, y\ [\mathsf{C}(x, y) \rightarrow \mathsf{C}(y, x)]$

Base RCC relations are defined in Table B.2. The two most studied RCC calculi are the so-called RCC-5 and RCC-8. The base relations of RCC-5 are EQ, DR, PO, PP, PPi. The base relations of RCC-8 are EQ, DC, EC, PO, TPP, TPPi, NTPP, NTPPi. The composition tables of RCC-5 and RCC-8 can be found in (Bennett, 1997).

Relation	Interpretation	Definition
$\mathsf{DC}(x, y)$	*x is disconnected from y*	$\neg\mathsf{C}(x, y)$
$\mathsf{P}(x, y)$	*x is a part of y*	$\forall z\ [\mathsf{C}(z, x) \rightarrow \mathsf{C}(z, y)]$
$\mathsf{PP}(x, y)$ $\mathsf{PPi}(y, x)$	*x is a proper part of y*	$\mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x)$
$\mathsf{EQ}(x, y)$	*x is identical with y*	$\mathsf{P}(x, y) \wedge \mathsf{P}(y, x)$
$\mathsf{O}(x, y)$	*x overlaps y*	$\exists z\ [\mathsf{P}(z, x) \wedge \mathsf{P}(z, y)]$
$\mathsf{DR}(x, y)$	*x is descrete from y*	$\neg\mathsf{O}(x, y)$
$\mathsf{PO}(x, y)$	*x partially overlaps y*	$\mathsf{O}(x, y) \wedge \neg\mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x)$
$\mathsf{EC}(x, y)$	*x is externally connected to y*	$\mathsf{C}(x, y) \wedge \neg\mathsf{O}(x, y)$
$\mathsf{TPP}(x, y)$ $\mathsf{TPPi}(y, x)$	*x is a tangential proper part of y*	$\mathsf{PP}(x, y) \wedge \exists z\ [\mathsf{EC}(z, x) \wedge \mathsf{EC}(z, y)]$
$\mathsf{NTPP}(x, y)$ $\mathsf{NTPPi}(y, x)$	*x is a nontangential proper part of y*	$\mathsf{PP}(x, y) \wedge \neg\exists z\ [\mathsf{EC}(z, x) \wedge \mathsf{EC}(z, y)]$

Table B.2: Definition of RCC relations.

B.3 Cardinal direction relations calculus

The language of cardinal direction relations (CDR) is introduced in (Skiadopoulos and Koubarakis, 2004). The universe is the set of regions of the Euclidean plane $\mathbb{R}^2$ with a coordinate system. The relational signature is a set of 3×3 binary matrices. A relation symbol $r = (r_{ij})$, where $i, j \in \{1, 2, 3\}$, corresponds to a binary relation R defined as follows. A pair of regions (A, B) belongs to R iff the intersection of A with (i, j)-th partition of space created by the bounding rectangle of B is empty if and only if $r_{ij} = 0$.

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