A Modal Logic for Observation-Based Knowledge Representation

Olivier Brunet
obrunet@inrialpes.fr

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Abstract

In this paper, we introduce and explore ways to include a notion of partiality of information in knowledge representation formalisms. This leads to the definition of an algebraic structure based on observation and partial representation, and to the study of the logical behaviour of those structures, with the characterization of a new modal logic called **OL**.

1 Introduction

In the process of the formalization of some system, any information available about its constitution or its state comes from observations which can be performed on the system. Thus, this notion of observation, and that of knowledge acquisition has to be taken into account, be considered when defining a model.

This idea is not new, and has already been studied [1, 9], focusing on the fact that it is only possible to have some finite information about the state of the system. This approach leads to the fact that only semi-decidable properties can be used for expressing informations about the system.

In the present article, we will study another approach. Instead of focusing on the finiteness of observations, we will rather take the fact that they have to be partial and imperfect as their most important property. Thus, we will use the notion of partial description as a formalization of the state of the system. Those descriptions can be embedded in a partially ordered set, a *representation*, where the informational content is used to define the ordering. We will also considerer the existence of severals points of observation of the system (and several representations). As all those representations are related to the same system, it is possible to exhibit some correlations between the descriptions appearing in different representations. Thus, in this context, we will replace the study of the behaviour of the system by the study of the relationships that exist between the different representations, and will forget the existence of the system, leading to the definition of a modal logic which formalizes the way partial information behaves.

In the following, we will introduce two algebraic structures : *representations* and *representations systems* which contain several representations and some ways to relate them. Then, we will define a logic on those structures, and characterize its behaviour, leading to the definition of the observational logic $\mathbf{OL}$. Finally, we will discuss some properties of this logic and show that there exists very weak relations between the knowledge accessible from different points of view.

2 Knowledge Representation Formalism

Let us first discuss what approach we want to take in order to formalize data. Any piece of information about the studied system can be seen as a partial description of its state. In the following, we will call such a piece of information a *description*, so that we only want to take into account the elements of information which are related to the state of the system, and forget anything about the physical implementation of the description. Another important notion which comes with descriptions, is that of comparison. As the descriptions we consider are partial, it follows that it is possible to compare them, by telling whether a description is more precise than another. This comparison relation acts as a partial order on descriptions, and we will include it in our framework, so that we get the following definition :

Definition 1
A representation is a pair $\mathcal{R} = \langle R, \leq \rangle$ where R is a set of descriptions and $\leq$ is a partial order on the descriptions.

Conventionally, if two descriptions are comparable then the smallest element corresponds to a more accurate description than the other : if $d_1 \leq d_2$, d_1 is the most accurate, and d_2 can be seen as an approximation of d_1.

With this definition, it appears that a representation is nothing more than a *poset*. Actually, many refinements of the definition could be considered, such as the existence of extremal elements or of internal operations such as the meet and the join. But since they shall play no role in the logical study we want to make, we will keep the previous definition unchanged, and use the terminology of *representation* and *description* instead of poset and element only in order to keep in mind the fact that they are related to an observation and a formal representation of a system. For notation convenience, we will always identify a representation with its set of descriptions.

2.1 Representation System

2.1.1 A collection of representations

A representation, as we have just defined it, is a set of elements which correspond to the descriptions that can possibly be made from a given point of view. But in the presence of partial observations, one should more generally consider that there exist several points of view. For instance, different observations may correspond to different resolutions. If an observation yields information on a determined part of the system, one can imagine a point of view observing another part of it. Many other possibilities may exist. The main reason for this lies in the word “partial” itself : knowing that a point of view is partial implies that there exists some information that is not available or accessible from this point of view, so that there might exist other points of view which permit to have access to those informations.

As a representation is associated to a point of view, one should in general consider the existence of several representations. In the following, let $\mathcal{I}$ be a set of indexes corresponding to the existing points of view, and for each $i \in \mathcal{I}$, let $\mathcal{R}_i$ be the representation associated to the observation process indexed by i.

2.1.2 A set of transformation functions

Since all the representations are supposed to be a formalization of a unique system, there should in general exist relations between the different representations. We will define a collection of functions between the representations to express this relationship. Here is the idea : suppose one has a description $d_i \in \mathcal{R}_i$ corresponding to the point of view i. This description correspond to some information about the state of the system. This implies pieces of information which can be accessible from another point of view $\mathcal{R}_j$ (possibly including no information at all). Let d_j be the best description of $\mathcal{R}_j$ corresponding to this and define $f_{j|i}(d_i) = d_j$.

This way, it is possible to define a collection of description transformation functions $f_{j|i} : \mathcal{R}_i \to \mathcal{R}_j$. If the previous considerations are rather informal, they allow us to give a characterization of those functions. First of all, the function which has the same representation as range and domain has to be the identity function on this representation, since the best description of a given state remains itself : $\forall\, d,\, f_{i|i}(d) = d$. Moreover, the $f_{j|i}$'s have to be monotonous : if one has two description $d_1 \leq d_2$ in $\mathcal{R}_i$ (this inequality means that d_1 represents more information that d_2), then any piece of information present in $f_{j|i}(d_2)$ has to be present in $f_{j|i}(d_1)$.

Finally, from the definition of the $f_{i|j}$'s, one may conclude that a description $d \in \mathcal{R}_i$ corresponds to more information that $f_{j|i}(d)$, even though they do not belong to the same representation. It is possible to express this fact using a third representation, and translate those two descriptions into this third representation. This leads to the following : let i, j, k be three indexes, and let d_k be a description of $\mathcal{R}_k$. If $f_{j|k}(d_k)$ has fewer information that d_k, then it implies that $f_{i|j} \circ f_{j|k}(d_k)$ also has fewer information than $f_{i|k}(d_k)$. This can be seen as a generalization of the monotony condition. This can be expressed by the following inequality :

$$\forall\, i, j, k \in \mathcal{I}, \forall\, d \in \mathcal{R}_k,\, f_{i|k}(d) \leq f_{i|j} \circ f_{j|k}(d) \tag{1}$$

We think that those conditions provide a good characterization of what a set of transformation functions should verify. They actually express very strong constraints on the different observations. First, they imply that they are all related to the same system. But more important, the last condition also means that they all correspond to observations of the system in a single state, since it also provides a way to relate the results of the different observations.

We can now combine those two elements together, in order to define the main algebraic structure which will be used to formalize and study observed systems.

Definition 2
A representation system *is a tuple :*

$$\mathcal{S} = \left\langle \mathcal{I}, \{\mathcal{R}_i\}_{i\in\mathcal{I}}, \{f_{i|j}\}_{i,j\in\mathcal{I}} \right\rangle$$

where $\mathcal{I}$ is a set of indexes, for each $i \in \mathcal{I}$, $\mathcal{R}_i$ is a representation, and for any $i, j \in \mathcal{I}$, $f_{i|j}$ is a monotonous function from $\mathcal{R}_j$ to $\mathcal{R}_i$. Moreover, those functions verify :

$$\forall\, i \in \mathcal{I},\, f_{i|i} = \mathrm{id}_i$$
$$\forall\, i, j, k \in \mathcal{I},\, f_{i|k} \leq f_{i|j} \circ f_{j|k}$$

Using this structure, we will now provide some ways to explore and characterize the logical behaviors which comes from the observational approach we are using.

3 Logical Characterization

In order to express facts about the system which is studied, we are now going to define a propositional language which sentences will be interpreted as assertions about the state of the system. Let Ψ denote a countable set of atomic propositions about the system. We define $\mathcal{L}_\Psi$ as the smallest language including Ψ, the false element $\perp$ (we assume that $\perp \notin \Psi$), and closed for $\vee$, $\wedge$, $\rightarrow$. We define $\neg\varphi$ as a short-cut for $\varphi \rightarrow \perp$, but as we will soon show, the logics we will exhibit are based on intuitionnistic logic, so that we cannot define $\wedge$ as a combination of $\vee$ and $\neg$.

We also need to define $\mathcal{L}_{\mathcal{I},\Psi}$, which also contains a set of modal operators $\{K_i\}_{i\in\mathcal{I}}$. Those operators will be used to express facts such that a given assertion can be proved using observations made from point of view i. This use of modal operators is similar to that in the field of epistemic logic [5].

3.1 Single Representations

In this section, we will first focus of the use of a single representation as the interpretation of logical assertions. Since we are considering only one point of view, we shall restrict ourselves to $\mathcal{L}_\Psi$, that is we do not consider the modal operators. Let $\mathcal{R}$ be a representation. Our first objective is to define an interpretation function $[\![\cdot]\!]$ which associates a sentence $\varphi \in \mathcal{L}_\Psi$ to a set of descriptions. More precisely, we define $[\![\varphi]\!]$ as the set of descriptions which provide enough information about the state of the system for proving that φ holds.

First, suppose that one has two descriptions $d_1 \leq d_2$, and that $d_2 \in [\![\varphi]\!]$. From the definition, d_1 provides more information than d_2. But if d_2 suffices to prove that φ holds, then so does d_1. This implies that for any $\varphi \in \mathcal{L}_\Psi$, its interpretation $[\![\varphi]\!]$ is an ideal of $\mathcal{R}$, that is a downward-closed subset (verifying $\forall\, d_1 \leq d_2,\, d_2 \in [\![\varphi]\!] \Rightarrow d_1 \in [\![\varphi]\!]$). If $\wp^{\downarrow}(P)$ denotes the set of ideals of a poset P, then $[\![\cdot]\!]$ is a function from $\mathcal{L}_\Psi$ to $\wp^{\downarrow}(\mathcal{R})$. We will now define this function inductively considering the structure of the terms. For atomic propositions, one has to provide their interpretations, so that we need

$$
\begin{aligned}
\forall\, \psi \in \Psi,\, [\![\psi]\!]_{\mathcal{R},\nu} &= \nu(\Psi)\\
[\![\varphi \vee \psi]\!]_{\mathcal{R},\nu} &= [\![\varphi]\!]_{\mathcal{R},\nu} \cup [\![\psi]\!]_{\mathcal{R},\nu}\\
[\![\varphi \wedge \psi]\!]_{\mathcal{R},\nu} &= [\![\varphi]\!]_{\mathcal{R},\nu} \cap [\![\psi]\!]_{\mathcal{R},\nu}\\
[\![\varphi \to \psi]\!]_{\mathcal{R},\nu} &= \{\, d \mid \forall\, d' \leq d,\, d' \in [\![\varphi]\!]_{\mathcal{R},\nu} \Rightarrow d' \in [\![\psi]\!]_{\mathcal{R},\nu}\}\\
[\![\neg\varphi]\!]_{\mathcal{R},\nu} &= \{\, d \mid \forall\, d' \leq d,\, d' \notin [\![\varphi]\!]_{\mathcal{R},\nu}\}\\
[\![\bot]\!]_{\mathcal{R},\nu} &= \emptyset
\end{aligned}
$$

Figure 1: Definition of $[\![\cdot]\!]_{\mathcal{R},\nu} : \mathcal{L}_\Psi \to \wp^{\downarrow}(\mathcal{R})$

$$\varphi \to \varphi \wedge \varphi \qquad \varphi \wedge \psi \to \psi \wedge \varphi \qquad (\varphi \to \psi) \to (\varphi \wedge \vartheta) \to (\psi \wedge \vartheta)$$

$$\varphi \to \varphi \vee \psi \qquad \varphi \vee \psi \to \psi \vee \varphi \qquad (\varphi \to \vartheta) \wedge (\psi \to \vartheta) \to (\varphi \vee \psi) \to \vartheta$$

$$\varphi \to (\psi \to \varphi) \qquad \varphi \wedge (\varphi \to \psi) \to \psi \qquad \bot \to \varphi$$

$$\frac{}{\vdash \varphi}\,\mathbf{Ax} \qquad \frac{\vdash \varphi \to \psi \quad \vdash \varphi}{\vdash \psi}\,\mathbf{M.P.}$$

Figure 2: Axioms and rules of intuitionnistic logic

an atomic interpretation function $\nu : \Psi \to \wp^{\downarrow}(\mathcal{R})$. The interpretation of the conjunction and the disjunction are straightforward, taken from their set-theoretic equivalent. For instance, $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$. It is not possible to do so for the implication $\to$: it would be tempting to define $[\![\varphi \to \psi]\!]$ as the set $\{\, d \mid d \in [\![\varphi]\!] \Rightarrow d \in [\![\psi]\!]\}$, but it is generally not an ideal. Instead, one has to restrict to the greatest ideal included in that set. We summarize this definition in figure 1.

With this function, we can define a validation symbol as follows : given a representation $\mathcal{R}$, an atomic interpretation function $\nu : \Psi \to \wp^{\downarrow}(\mathcal{R})$, and a description $d \in \mathcal{R}$, we say that a proposition $\varphi \in \mathcal{L}_\Psi$ is valid for d and write $\mathcal{R}, \nu, d \models_R \varphi$ if and only if $d \in [\![\varphi]\!]_{\mathcal{R},\nu}$. We also define $\mathcal{R}, \nu \models_R \varphi$ for $[\![\varphi]\!]_{\mathcal{R},\nu} = \mathcal{R}$ and $\models_R \varphi$ if and only if $\mathcal{R}, \nu \models_R \varphi$ for all representations $\mathcal{R}$ and atomic interpretations $\nu : \Psi \to \wp^{\downarrow}(\mathcal{R})$.

In the next proposition, we will show that this validation symbol $\models_R$ is exactly equivalent to intuitionnistic logic. This logic, which formulation is given in figure 2, was defined as an alternative to classical logic, where the excluded middle principle is rejected, and where implication embodies a cause to consequence relation much stronger that its equivalent in classical logic. Let us write $\vdash_{\mathbf{IL}} \varphi$ to denote that φ can be proved using the axioms of intuitionnistic logic.

Proposition 1
Given a sentence $\varphi \in \mathcal{L}_\Psi$, one has the equivalence :

$$\vdash_{\mathbf{IL}} \varphi \quad \Leftrightarrow \quad \models_R \varphi$$

Proof This comes from the fact that representation, which are just posets, can be seen as Kripke structures, where the accessibility relation is reflexive and transitive. Such structures are the usual models for intuitionnistic logic [6, 4]. □

Now that we have precisely characterized the logic associated to representations, we will turn to representation systems. This time, there will be several points of view to consider, so that it is the full language $\mathcal{L}_{\mathcal{I},\Psi}$ which will be used now.

3.2 General Representation Systems

3.2.1 Extension of the interpretation function

We will now consider a representation system $\mathcal{S} = \left\langle \mathcal{I}, \{\mathcal{R}_i\}, \left\{f_{i|j}\right\} \right\rangle$. We have defined an interpretation function $[\![\cdot]\!]$ associated to a given representation. Since a representation system is made of several representation, one will have to consider as many interpretation functions $[\![\cdot]\!]_i : \mathcal{L}_{\mathcal{I},\Psi} \to \wp^{\downarrow}(\mathcal{R}_i)$. Moreover, as an interpretation function is based on an atomic interpretation function (previously denoted ν), we will also need to consider a collection of atomic interpretations $\nu_i : \Psi \to \wp^{\downarrow}(\mathcal{R}_i)$ in order to define the $[\![\cdot]\!]_i$.

We also need to define the interpretation of the modal K_i operators. For doing this, consider $d \in [\![K_i\,\varphi]\!]_j$ (in particular, $d \in \mathcal{R}_j$). From the interpretation of K_i, this means that d corresponds to a description which contains enough information to assert that the property φ in the representation $\mathcal{R}_i$. Since the description d belongs to $\mathcal{R}_j$, it follows that if it is transformed into an $\mathcal{R}_j$ description using $f_{i|j}$, it is included in the interpretation of φ for i. Stated formally, one has :

$$d \in [\![K_i\,\varphi]\!]_j \;\;\Leftrightarrow\;\; f_{i|j}(d) \in [\![\varphi]\!]_i$$

This leads to the definition of $\{[\![\cdot]\!]_{\mathcal{S},\nu,i}\}_{i\in\mathcal{I}}$ given in figure 3, where $\mathcal{S}$ is a representation system, and ν denotes the collection $\left\{\nu_i : \Psi \to \wp^{\downarrow}(\mathcal{R}_i)\right\}$. We define the notation $\models_S \varphi$ which means that for any representation system $\mathcal{S}$, and for any atomic interpretation functions $\nu = \{\nu_i\}_{i\in\mathcal{I}}$, one has :

$$\forall\, i \in \mathcal{I},\, [\![\varphi]\!]_{\mathcal{S},\nu,i} = \mathcal{R}_i$$

Before starting the characterization of the logic which corresponds to $\models_S$, we first give two useful propositions. The first one, which concerns intuitionnistic logic, gives a simpler version of the validation of an implication.

Proposition 2
Given two terms φ and ψ, one has :

$$\models_S \varphi \to \psi \;\Leftrightarrow\; \forall\,\mathcal{S}, \nu, i,\, [\![\varphi]\!]_{\mathcal{S},\nu,i} \subseteq [\![\psi]\!]_{\mathcal{S},\nu,i}$$

$$
\begin{aligned}
\forall\, \psi \in \Psi,\, [\![\psi]\!]_{\mathcal{S},\nu,i} &= \nu_i(\Psi) \\
[\![\varphi \vee \psi]\!]_{\mathcal{S},\nu,i} &= [\![\varphi]\!]_{\mathcal{S},\nu,i} \cup [\![\psi]\!]_{\mathcal{S},\nu,i} \\
[\![\varphi \wedge \psi]\!]_{\mathcal{S},\nu,i} &= [\![\varphi]\!]_{\mathcal{S},\nu,i} \cap [\![\psi]\!]_{\mathcal{S},\nu,i} \\
[\![\varphi \to \psi]\!]_{\mathcal{S},\nu,i} &= \{\, d \mid \forall\, d' \leq d,\, d' \in [\![\varphi]\!]_{\mathcal{S},\nu,i} \Rightarrow d' \in [\![\psi]\!]_{\mathcal{S},\nu,i} \} \\
[\![\bot]\!]_{\mathcal{S},\nu,i} &= \emptyset \\
[\![K_j\, \varphi]\!]_{\mathcal{S},\nu,i} &= \left\{\, d \;\middle|\; f_{j|i}(d) \in [\![\varphi]\!]_{\mathcal{S},\nu,j} \right\}
\end{aligned}
$$

Figure 3: Definition of $\left\{[\![\cdot]\!]_{\mathcal{S},\nu,i} : \mathcal{L}_\Psi \to \wp^{\downarrow}(\mathcal{R}_i)\right\}_{i\in\mathcal{I}}$

Proof It comes from the following simple calculation :

$$
\begin{aligned}
\models_S \varphi \to \psi \Leftrightarrow\ & \forall\, \mathcal{S}, \nu, i,\, [\![\varphi \to \psi]\!]_{\mathcal{S},\nu,i} = \mathcal{R}_i \\
\Leftrightarrow\ & \forall\, \mathcal{S}, \nu, i, \forall\, d \in \mathcal{R}_i,\, d \in [\![\varphi]\!]_{\mathcal{S},\nu,i} \Rightarrow d \in [\![\psi]\!]_{\mathcal{S},\nu,i} \\
\Leftrightarrow\ & \forall\, \mathcal{S}, \nu, i,\, [\![\varphi]\!]_{\mathcal{S},\nu,i} \subseteq [\![\psi]\!]_{\mathcal{S},\nu,i}
\end{aligned}
$$

□

The second one relates, for a given index i, the modal operator K_i and the interpretation $[\![\cdot]\!]_i$.

Proposition 3
Given an index $i \in \mathcal{I}$ and a term φ, one has :

$$[\![K_i\, \varphi]\!]_i = [\![\varphi]\!]_i$$

Proof It comes from the fact that $f_{i|i}$ is the identity on $\mathcal{R}_i$, so that :

$$[\![K_i\, \varphi]\!]_i = \left\{\, d \;\middle|\; f_{i|i}(d) \in [\![\varphi]\!]_i \right\} = \{\, d \mid d \in [\![\varphi]\!]_i \} = [\![\varphi]\!]_i$$

□

3.2.2 Identification of the corresponding logic

Since the definition of the interpretation $[\![\cdot]\!]_{\mathcal{S}}$ is an adaptation of the previous interpretation defined on single representations, it follows that the logic modelized by representations systems will be based on intuitionnistic logic.

Some axioms have to be added to it in order to give a formalization of the behaviour of the modal operators, so that we will now study the validation of modal axioms. First, we will consider some classic axioms taken from the literature [3, 8] : we will explore the validation of $\mathbf{K}$, $\mathbf{D}$, $\mathbf{T}$, $\mathbf{4}$ and $\mathbf{5}$.

The first axiom, $\mathbf{K} : K_i\,(\varphi \to \psi) \to K_i\,\varphi \to K_i\,\psi$ can be interpreted in the present situation as the fact that an agent $\mathcal{A}_i$ associated to a given point of view (and a representation $\mathcal{R}_i$) can make deduction. In other words, if $\mathcal{A}_i$ has enough information for proving that both $\varphi \to \psi$ and φ hold for the observed system, then ψ also holds. This can be proved, using proposition 2, by showing that $[\![K_i\,(\varphi \to \psi)]\!]_j \subseteq [\![K_i\,\varphi \to K_i\,\psi]\!]_j$. This is true, since :

$$[\![K_i\,(\varphi \to \psi)]\!]_j = \left\{ d \;\middle|\; \forall\, d' \leq f_{i|j}(d),\, d' \in [\![\varphi]\!]_i \Rightarrow d' \in [\![\psi]\!]_i \right\}$$
$$[\![K_i\,\varphi \to K_i\,\psi]\!]_j = \left\{ d \;\middle|\; \forall\, d' \leq d,\, f_{i|j}(d') \in [\![\varphi]\!]_i \Rightarrow f_{i|j}(d') \in [\![\psi]\!]_j \right\}$$

and using the monotony of $f_{i|j}$, one has $d' \leq d \Rightarrow f_{i|j}(d') \leq f_{i|j}(d)$.

The axiom $\mathbf{D} : K_i\,\varphi \to \neg K_i\,\neg\varphi$ is also valid for representation systems. Used in conjunction with $\mathbf{K}$, it is equivalent to $K_i\,\bot \to \bot$. It is easy to show that this latter is valid for representation systems, since :

$$[\![K_i\,\bot]\!]_j = \left\{ d \;\middle|\; f_{i|j}(d) \in [\![\bot]\!]_j \right\} = \left\{ d \;\middle|\; f_{i|j}(d) \in \emptyset \right\} = \emptyset = [\![\bot]\!]_j$$

Using the previous agent interpretation, this means that the knowledge of a given agent is consistent, since it cannot prove the absurd proposition. Yet, as we will now show, this does not imply that the knowledge of an agent corresponds to properties that are actually verified by the observed system. This is reflected by the fact that the axiom $\mathbf{T} : K_i\,\varphi \to \varphi$ is not valid. It is possible to show this by considering a representation system with at least two indices i and j, and a set of atomic interpretation function $\{\nu_i\}_i$ such that $\nu_i(\psi) = \mathcal{R}_i$ and $\nu_j(\psi) = \emptyset$ (where ψ is an atomic proposition). If $\mathbf{T}$ were verified, it would imply that $[\![K_i\,\psi]\!]_j \subseteq [\![\psi]\!]_j$. But in the present case, $[\![K_i\,\psi]\!]_j = \mathcal{R}_j$ and $[\![\psi]\!]_j = \emptyset$.

Actually, a weaker version of $\mathbf{T}$ is valid for representation systems. This version reduces the scope of $\mathbf{T}$ to proposition of the form $K_i\,\varphi$. More precisely, we define the axiom $\mathbf{T}_2 : K_i\,K_j\,\varphi \to K_j\,\varphi$. Its validity is the exact expression that the transformation functions verify $f_{i|k} \leq f_{i|j} \circ f_{j|k}$, since :

$$\begin{aligned} & [\![K_i\,K_j\,\varphi]\!]_k \subseteq [\![K_j\,\varphi]\!]_k \\ \Leftrightarrow\ & f_{j|i} \circ f_{i|k}(d) \in [\![\varphi]\!]_j \Rightarrow f_{j|k}(d) \in [\![\varphi]\!]_j \\ \Leftrightarrow\ & f_{j|k}(d) \leq f_{j|i} \circ f_{i|k}(d) \end{aligned}$$

Thus, in this formalism, the knowledge an agent has about the studied system might not correspond to reality, but it is consistent, and what she knows about the knowledge of another agent is exact.

We will now turn to the introspection axioms. First, let's consider $\mathbf{4} : K_i\,\varphi \to K_i\,K_i\,\varphi$. This axioms expresses the fact that an agent knows that she knows some particular proposition. It is valid for representation systems, since $f_{i|i}$ is the identity on $\mathcal{R}_i$. The other introspection axiom $\mathbf{5} : \neg K_i\,\varphi \to K_i\,\neg K_i\,\varphi$, which applies to things which are not known (or more precisely that cannot be known), is not valid. This can be understood from the postulate that knowledge comes from observation, since knowledge would in the case of $\mathbf{5}$ also come from an absence of observation.

Using proposition 3, it is possible to define an extra axiom which expresses more precisely how introspection occurs in the present framework. As shown above, one has :

$$[\![K_i\,\varphi]\!]_i = [\![\varphi]\!]_i$$

It follows from this that if we define $\mathbf{L}$: $K_i\ (\varphi \leftrightarrow K_i\,\varphi)$, this axiom is valid for representation systems. Since the equivalence $\leftrightarrow$ is only a notation and not a real connector, this axiom can be split into two parts :

$$\mathbf{L_T} : K_i\ (K_i\,\varphi \rightarrow \varphi) \qquad \mathbf{L_4} : K_i\ (\varphi \rightarrow K_i\,\varphi)$$

With those notations, it is easy to see that with $\mathbf{K}$, $\mathbf{L_4}$ implies $\mathbf{4}$. Thus, we have given a characterization of an agent's knowledge about herself which is better than just stating that $\mathbf{4}$ is verified.

To finish this survey of axioms valid for representation systems, we will also introduce $\mathbf{KV}$: $K_i\,(\varphi \vee \psi) \rightarrow K_i\,\varphi \vee K_i\,\psi$. This axiom can be seen as an adaption to agents of the way the disjunction operation behaves in intuitionnistic logic. Thus, if an agents knows that $\varphi \vee \psi$ is true, then she knows that either φ or ψ is true.

Finally, let's take a look at the rules which can be used. First, the $\mathbf{Nec}$ (necessity) rule, which allows to infer $\vdash K_i\,\varphi$ from $\vdash \varphi$ is valid, since $[\![\varphi]\!]_i = \mathcal{R}_i$ implies that $[\![K_i\,\varphi]\!]_j = \mathcal{R}_j$. Another rule can be defined, which expresses the way $\models_S$ is defined. If a proposition φ is valid for representation systems, then it means that given a representation system $\mathcal{S}$, one has $\forall\, i$, $[\![\varphi]\!]_i = \mathcal{R}_i$. But this is equivalent to stating that $\forall\, i, j$, $[\![K_i\,\varphi]\!]_j = \mathcal{R}_j$. This leads to the definition of a new rule which we call $\mathbf{Univ}$ (universality) :

$$\frac{\forall\, i \in \mathcal{I},\ \vdash K_i\,\varphi}{\vdash \varphi}\,\mathbf{Univ}$$

With all those definitions, we will now introduce the observational logic $\mathbf{OL}$. This logic is an intuitionnistic modal logic which modal axioms are $\mathbf{K}$, $\mathbf{D}$, $\mathbf{L}$, $\mathbf{T_2}$ and $\mathbf{KV}$ and which modal rules are $\mathbf{Nec}$ and $\mathbf{Univ}$. A summary is given in figure 4. This logic is exact the logic modeled by representations systems :

Proposition 4
Given a sentence $\varphi \in \mathcal{L}_{\mathcal{I},\Psi}$, one has the equivalence :

$$\vdash_{\mathbf{OL}} \varphi \quad \Leftrightarrow \quad \models_S \varphi$$

Proof The complete proof can be found in the appendix. □

It is interesting to remark that the logic we have just identified has strong relationships with $\mathbf{IS4} + \mathbf{KV}$. This logic has been identified in the case of representation systems where there exists a representation which is more expressive than the others, and where all knowledge is related to this representation [2]. Yet, $\mathbf{OL}$ is neither weaker

$$K_i\,(\varphi \to \psi) \to K_i\,\varphi \to K_i\,\psi \qquad \mathbf{K}$$

$$K_i\,\varphi \to \neg K_i\,\neg\varphi \qquad \mathbf{D}$$

$$K_i\;(\varphi \leftrightarrow K_i\,\varphi) \qquad \mathbf{L}$$

$$K_i\,K_j\,\varphi \to K_j\,\varphi \qquad \mathbf{T_2}$$

$$K_i\,(\varphi \vee \psi) \to K_i\,\varphi \vee K_i\,\psi \qquad \mathbf{KV}$$

$$\frac{\vdash \varphi}{\vdash K_i\,\varphi} \qquad \mathbf{Nec}$$

$$\frac{\forall\, i \in \mathcal{I},\ \vdash K_i\,\varphi}{\vdash \varphi} \qquad \mathbf{Univ}$$

Figure 4: Modal axioms and rules of $\mathbf{OL}$

nor stronger than $\mathbf{IS4}+\mathbf{KV}$. Rather, this logic is a variant of the latter, and the fact that it does not verify $\mathbf{T}$ leads to some interesting properties about the way the knowledge of different agents can be related.

4 Discussion

Using this framework and the logic we have devised, it is possible to initiate a discussion about the possible relationships that exist between the knowledge of different agents. In the definition of $\mathbf{OL}$, only one axiom does relate the existence of different modal operators. This axiom is the one we called $\mathbf{T_2}$:

$$\mathbf{T_2} : K_i\,K_j\,\varphi \to K_j\,\varphi$$

It expresses the fact that if an agent $\mathcal{A}_i$ knows that another agent $\mathcal{A}_j$ knows a property φ, then agent $\mathcal{A}_j$ actually knows φ. Yet, this axiom is weaker than $\mathbf{T}$, and does not permit to deduce any property about $\mathcal{A}_i$ knowing φ. This can be expressed by the fact that agents do not trust each other. If it were the case, it would mean that representation systems do verify an axiom which we call $\mathbf{C}$:

$$\mathbf{C} : K_i\,K_j\,\varphi \to K_i\,\varphi$$

This axiom is very similar to $\mathbf{T_2}$, except that it removes the rightmost modal operator instead of the leftmost. Stated another way, with $\mathbf{T_2}$, one has to keep the rightmost modal operator, and $\mathbf{C}$ would allow to remove it. It is easy to show that this axiom is not valid for representation systems, since it would imply the validity of $\mathbf{T}$, as shows the following derivation :

$$
\begin{aligned}
&\forall\, i, j, k,\ [\![K_i\, K_j\, \varphi]\!]_k \subseteq [\![K_i\, \varphi]\!]_k \\
\Rightarrow &\forall\, i, j, k,\ \left\{ d \;\middle|\; f_{i|k}(d) \in [\![K_j\, \varphi]\!]_i \right\} \subseteq \left\{ d \;\middle|\; f_{i|k}(d) \in [\![\varphi]\!]_i \right\} \\
\Rightarrow &\forall\, i, j,\ \left\{ d \;\middle|\; f_{i|i}(d) \in [\![K_j\, \varphi]\!]_i \right\} \subseteq \left\{ d \;\middle|\; f_{i|i}(d) \in [\![\varphi]\!]_i \right\} \\
\Rightarrow &\forall\, i, j,\ [\![K_j\, \varphi]\!]_i \subseteq [\![\varphi]\!]_i
\end{aligned}
$$

Thus, we have shown that agents cannot trust each other. A weaker way to relate the agents' knowledge can be expressed by a condition of global consistency, which we formalize by axiom $\mathbf{GD}$:

$$\mathbf{GD} : K_i\, \varphi \to \neg K_j\, \neg\varphi$$

It can be shown that this axiom is not valid either, and it is even possible to build a representation system which modelizes $K_i\, \varphi \wedge K_j\, \neg\varphi$. This implies that it is not even possible to ensure that for a defined state of the system, a property and its negation cannot be observed.

As we have seen, even though we have identified the observational logic $\mathbf{OL}$ from a very general algebraic structure used to formalize the notion of partial observation, this logic expresses very weak conditions on the relations between the knowledge of the different observers. More precisely, it appears that the only way to relate two different agents is the use of axiom $\mathbf{T_2}$ which, as mentioned earlier, forces to keep the rightmost modal operator. Thus, it is impossible to disconnect an observation from its originating point of view.

Yet, this does not imply that $\mathbf{OL}$ is useless, or that it cannot express any relation between agents. On the contrary, its behaviour shows that an important point which has to be taken into account is that as soon as the originating point of view is considering, thus leading to consider propositions of the form $K_i\, \varphi$, then the information is trustworthy, and the axioms $\mathbf{C}$ and $\mathbf{GD}$ are verified. For instance, one has :

$$\vdash_{\mathbf{OL}} K_i\, K_j\, K_k\, \varphi \to K_i\, K_k\, \varphi$$

Thus, propositions of this form can be used safely to describe the state of the system, since it does not depend on a specific representation. This suggests the existence of a category of propositions which behave the same way, and can be used safely.

5 Conclusion

In this article, we have introduced a simple algebraic structure in order to formalize knowledge and information based on partial and incomplete observation. This led us to the definition of *representation structures*, using very general assumptions on the structure of studied systems. Using those structures, we have defined a specific logic, namely $\mathbf{OL}$, intended to formalize the behaviour of information in such a context.

The study of this logic has permitted to identify some very interesting and maybe counter-intuitive properties of information. The first point is that it is based on intuitionnistic logic, rather than on classical logic. Thus, the excluded middle principle is not verified. Similarly, the modal axiom $\mathbf{5} : \neg K_i\,\varphi \rightarrow K_i\,\neg K_i\,\varphi$ is not verified either. This suggests that knowledge comes solely from observations, and is obtained inductively.

Another important aspect is that there exists a category of propositions which can be used without any reference to a particular representation, or point of view. We have seen that propositions of the form $K_i\,\varphi$ are in this category, since a particular representation is explicitly given. Yet, there might exist other propositions in this category, and this notion is still to be studied.

References

[1] S. Abramsky, *Domain Theory and the Logic of Observable Properties*, PhD Thesis, Queen Mary College, University of London, 1987

[2] O. Brunet, *A Model for Knowledge Representation in Distributed Systems*, Proc. KI workshop on Modal Logic in AI, Wien (OS), pages 3–12, 2001

[3] B. Chellas, *Modal Logic, an Introduction*, Cambridge University Press, 1980.

[4] D. van Dalen, *Intuitionnistic Logic*, in D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume III, D. Reidel Publishing, 1986

[5] R. Fagin, J. Y. Halpern, Y. Moses and M. Y. Vardi, *Reasoning about Knowledge*, The MIT Press, 1995

[6] M. Fitting, *Proof Methods for Modal and Intuitionnistic Logics*, vol. 169, D. Reidel Publishing, 1983

[7] A. Heyting, *Intuitionism: An Introduction*, North-Holland, 1956

[8] G.E. Hughes and M. J. Cresswell, *A New Introduction to Modal Logic*, Routledge, 1996

[9] S. Vickers, *Topology via Logics* Number 6 in Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 1989

A Proof of completeness

We will prove the completeness by defining a canonical model of $\mathbf{OL}$. For this, let us define for each index $i \in \mathcal{I}$ the representation $\mathcal{R}_i$. Let this representation be the set of all prime sets Γ_i of $\mathbf{O}L$ such that they contain $\varphi \leftrightarrow K_i\,\varphi$ for all propositions φ, ordered by reverse inclusion. We recall that given a logic $\mathcal{L}$, a set Γ is prime for this logic if and only if it is closed by deduction for $\mathbf{L}$, it is consistent (so that $\perp \notin \Gamma$), and if $\varphi \vee \psi \in \Gamma$, then either φ or ψ belongs to Γ. This is a simple adaption of the canonical model used for intuitionnistic logic as it can be found in [4] for instance.

We also define $f_{i|j}(\Gamma_j) = \{\varphi \mid K_i\,\varphi \in \Gamma_j\}$ and $\nu_i(\psi) = \{\Gamma_i \mid \psi \in \Gamma_i\}$. With those definitions, it is easy to check that $\mathcal{S}_C = \left\langle \mathcal{I}, \{\mathcal{R}_i\}, \left\{f_{i|j}\right\}\right\rangle$ is a representation system. The canonical interpretation $[\![\cdot]\!]_C$ is defined as $[\![\cdot]\!]_{\mathcal{S}_C,\nu}$. This interpretation verifies :

Proposition 5
For any proposition φ, one has :

$$\forall\, i,\, [\![\varphi]\!]_{C,i} = \{\, \Gamma_i \mid \varphi \in \Gamma_i \}$$

Proof This result is proved by induction on the length of the considered term. We will only develop the modal case :

$$\begin{aligned}
[\![K_i\, \varphi]\!]_{C,j} &= \left\{\, \Gamma_j \;\middle|\; f_{i|j}(\Gamma_j) \in [\![\varphi]\!]_{C,i} \right\} \\
&= \left\{\, \Gamma_j \;\middle|\; \varphi \in f_{i|j}(\Gamma_j) \right\} \\
&= \{\, \Gamma_j \mid K_i\, \varphi \in \Gamma_j \}
\end{aligned}$$

$\square$

Proposition 6
One has :

$$[\![\varphi]\!]_{C,i} = \mathcal{R}_i \Rightarrow \vdash_{\mathbf{OL}} K_i\, \varphi$$

Proof If $[\![\varphi]\!]_{C,i} = \mathcal{R}_i$, then there exists a finite set $\{\psi_1, \ldots, \psi_n\}$ such that :

$$\vdash_{\mathbf{OL}} (\psi_1 \leftrightarrow K_i\, \psi_1) \wedge \ldots \wedge (\psi_n \leftrightarrow K_i\, \psi_n) \rightarrow \varphi$$

Using rule $\mathbf{Nec}$, this implies :

$$\vdash_{\mathbf{OL}} K_i\, (\psi_1 \leftrightarrow K_i\, \psi_1) \wedge \ldots \wedge K_i\, (\psi_n \leftrightarrow K_i\, \psi_n) \rightarrow K_i\, \varphi$$

So that one can deduce $\vdash_{\mathbf{OL}} K_i\, \varphi$ thanks to axiom **L**. $\square$

Proposition 7
For all $\varphi \in \mathcal{L}_{\mathcal{I},\Psi}$, one has :

$$\vdash_{\mathbf{OL}} \varphi \Leftrightarrow\, \models_S \varphi$$

Proof The soundness direction can easily be checked by hand, and has been sketched in 3.2.2. The completeness direction

$$\models_S \varphi \Rightarrow\; \forall\, i,\, [\![\varphi]\!]_{C,i} = \mathcal{R}_i \Rightarrow\; \forall\, i,\; \vdash_{\mathbf{OL}} K_i\, \varphi \Rightarrow\, \vdash_{\mathbf{OL}} \varphi$$

The last implication comes from the application of the universality rule. $\square$